计算机科学与工程学院（网络空间安全学院）

Algorithmic Game Theory

算法博弈论

张彦如

2019 Fall

Course Content and Calendar

Content

1. Introduction and Examples
2. Mechanism Design Basics
3. Myerson's Lemma
4. Algorithmic Mechanism Design
5. Selfish Routing and the Price of Anarchy
6. Over-Provisioning and Atomic Selfish Routing

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Over-Provisioning and Atomic Selfish Routing

Lecture 6

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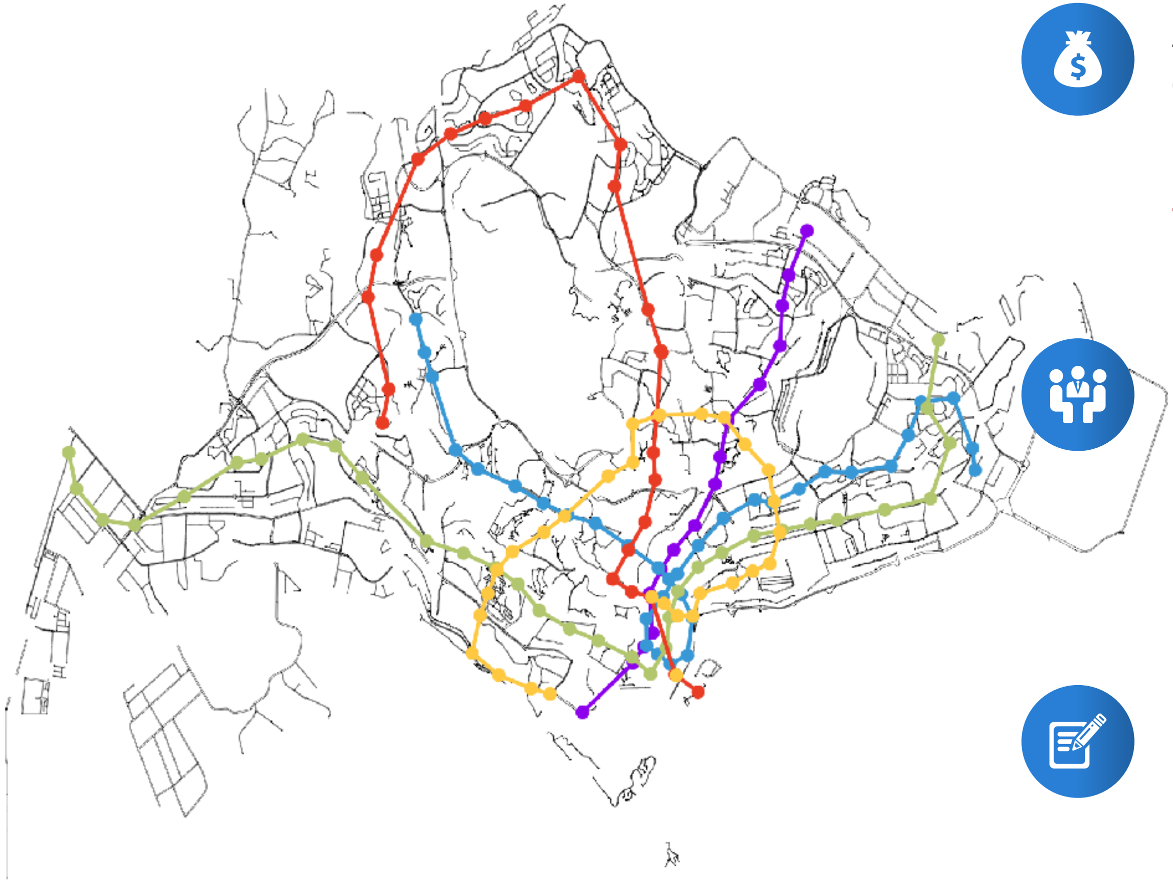
Motivation



Applications

The selfish routing model introduced last lecture can provide insight into many different kinds of networks, including transportation, communication, and electrical networks

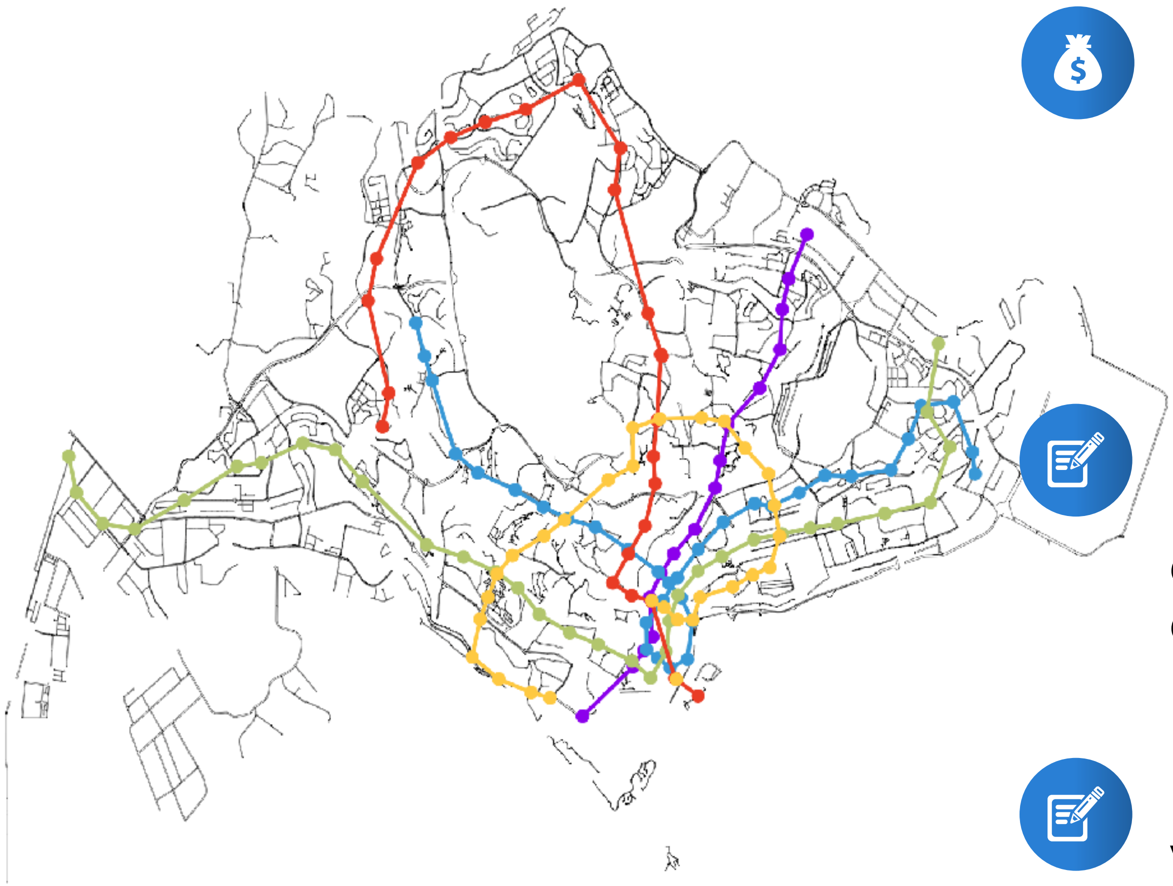
Communication Networks

One big advantage in communication networks is that it's often relatively cheap to add additional capacity to a network

Network Management

Because of this, a popular strategy to communication network management is to install more capacity than is needed, meaning that the network will generally not be close to fully utilized

Motivation



Reasons for Network Over-Provisioning

- Anticipate future growth in demand
- It has been observed empirically that networks tend to perform better — for example, suffering fewer packet drops and delays — when they have extra capacity

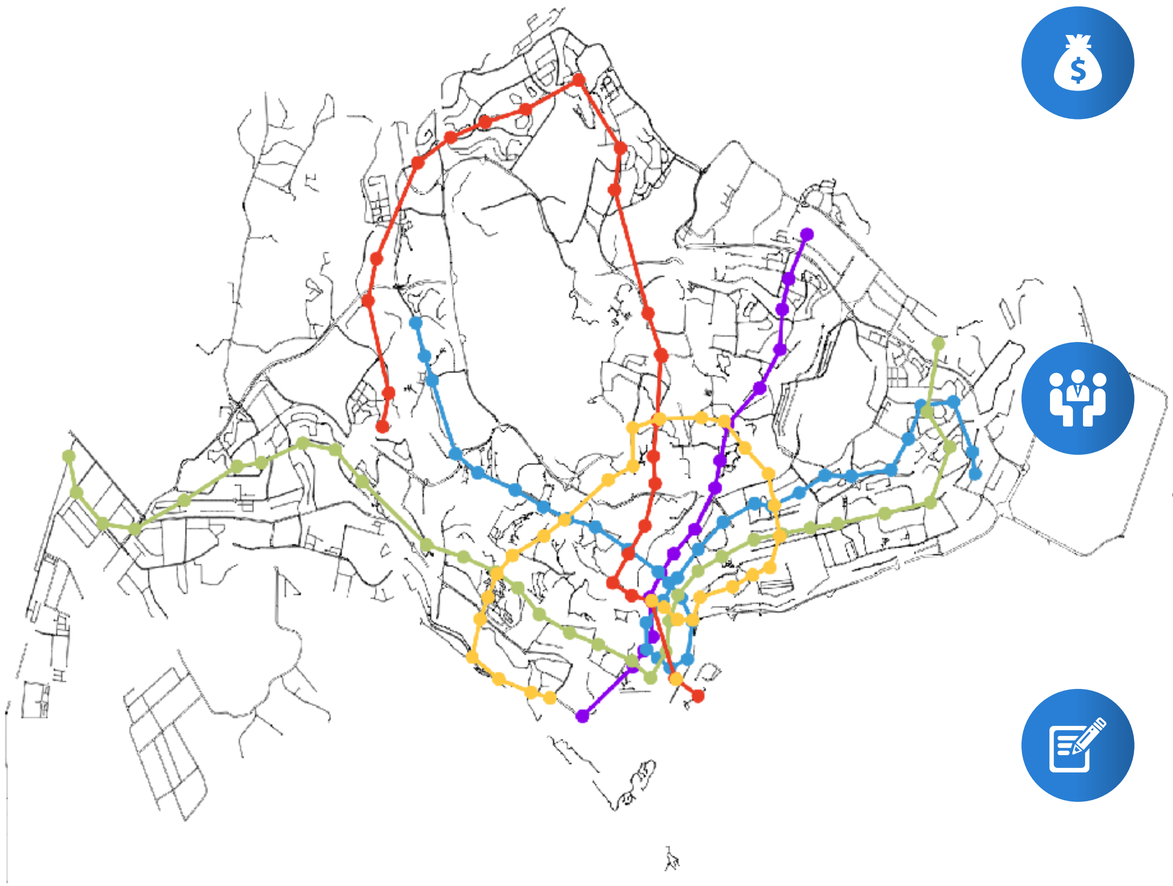
“Quality-of-Service (QoS)” Guarantees

Network over-provisioning has been used as an alternative to directly enforcing “quality-of-service (QoS)” guarantees (e.g., delay bounds

For Example

Via an admission control protocol that refuses entry to new traffic when too much congestion would result

Motivation



The Goal of This Section

Develop theory to corroborate the empirical observation that network over-provisioning leads to good performance

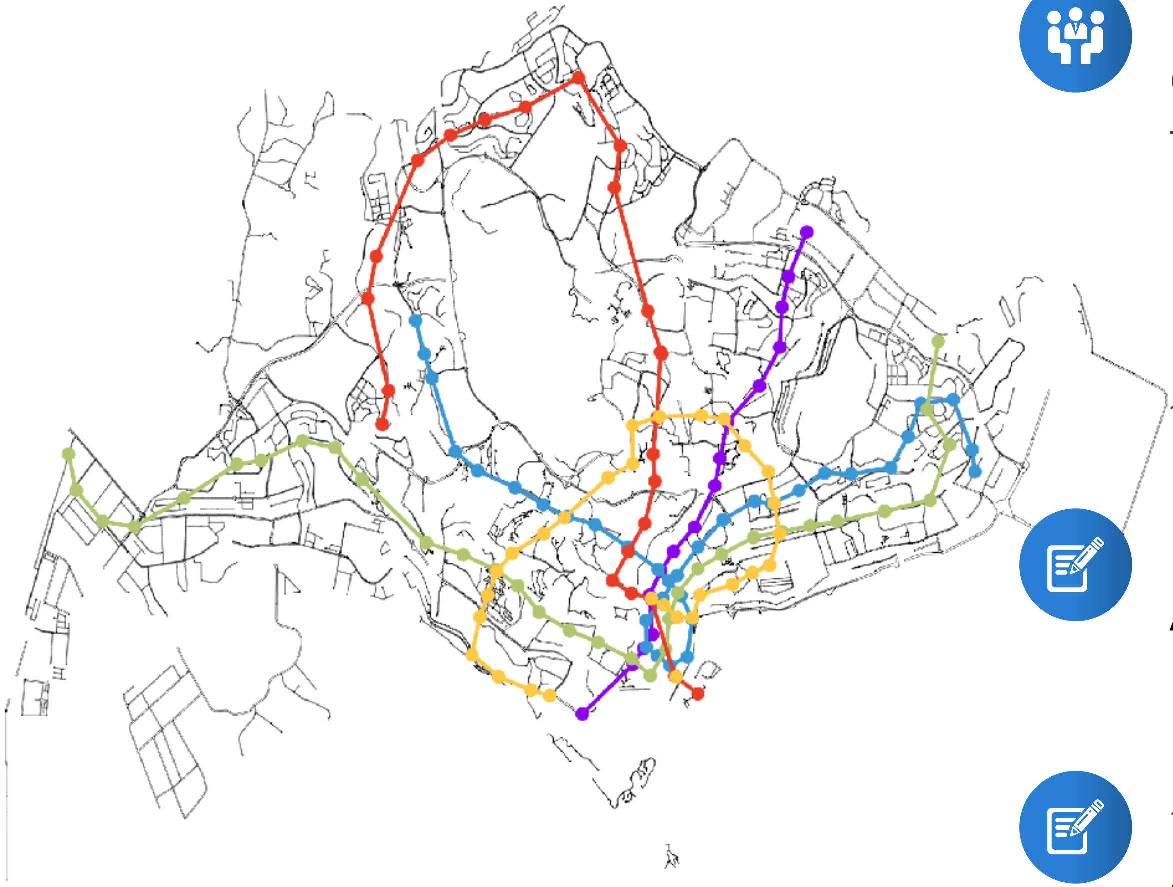
Section 1.2

Shows how to apply directly the theory developed last lecture to over-provisioned networks

Section 1.3

Offers a second approach to proving the same point, that selfish routing with extra capacity is competitive with optimal routing

POA Bounds for Over-Provisioned Networks



Network Model

Consider a network in which every cost function $c_e(x)$ has the form

$$c_e(x) = \begin{cases} \dfrac{1}{u_e - 1} & if\ x < u_e \\ +\infty & if\ x \geq u_e \end{cases}$$

u_e should be thought of as the capacity of edge e

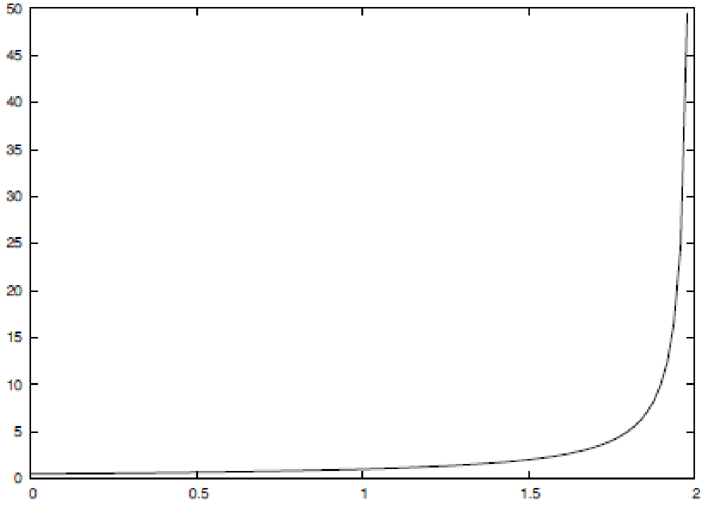
Expected Delay in $M/M/1$ Queue

A cost function of the form is the expected per-unit delay in an M/M/1 queue

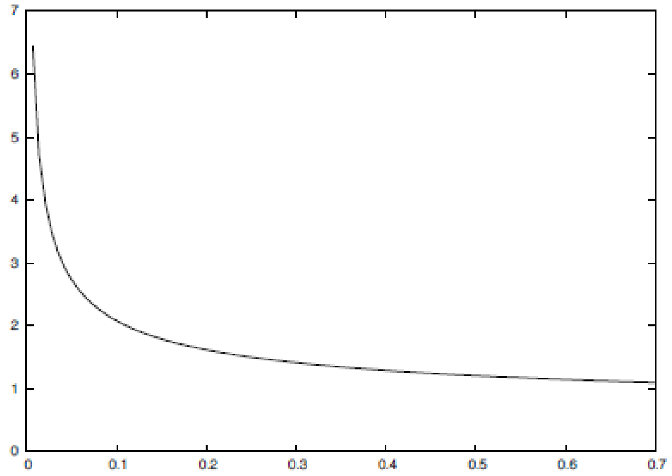
$M/M/1$ Queue

A queue where jobs arrive according to a Poisson process with rate x and have independent and exponentially distributed services times with mean $1/u_e$

POA Bounds for Over-Provisioned Networks



(a) M/M/1 delay function



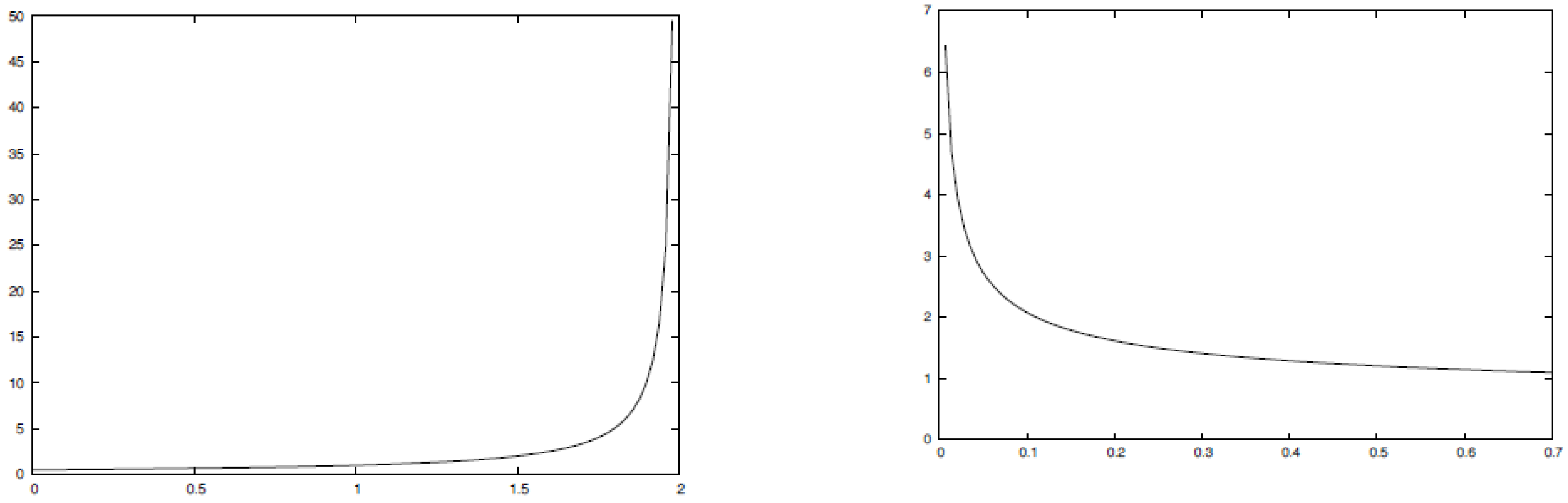
(b) Extra capacity vs. POA curve

Figure 1: Modest overprovisioning guarantees near-optimal routing. The left-hand figure displays the per-unit cost $c(x) = 1/(u - x)$ as a function of the load x for an edge with capacity $u = 2$. The right-hand figure shows the worst-case price of anarchy as a function of the fraction of unused network capacity.

Figure 1

- Figure 1(a) displays such a function; it stays very flat until the traffic load nears the capacity, at which point the cost rapidly tends to $+\infty$
- This is the simplest cost function used to model delays in communication networks

POA Bounds for Over-Provisioned Networks



(a) M/M/1 delay function

(b) Extra capacity vs. POA curve

Figure 1: Modest overprovisioning guarantees near-optimal routing. The left-hand figure displays the per-unit cost $c(x) = 1/(u - x)$ as a function of the load x for an edge with capacity $u = 2$. The right-hand figure shows the worst-case price of anarchy as a function of the fraction of unused network capacity.

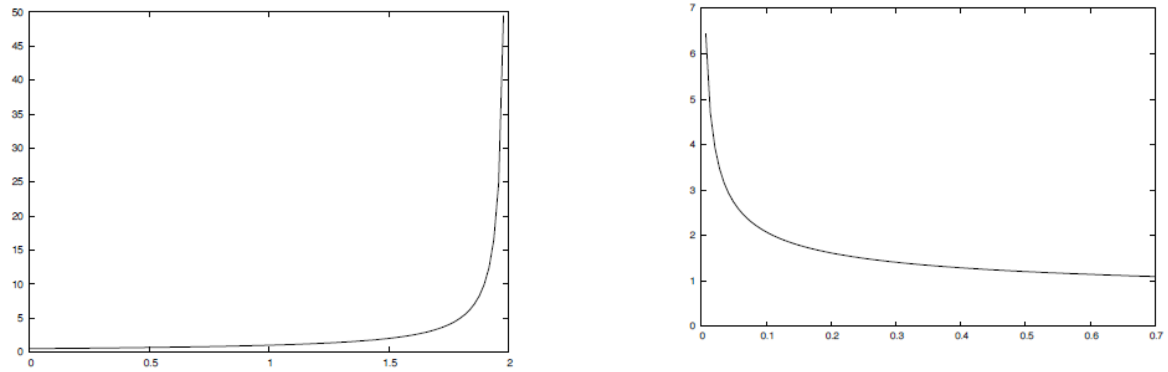
β-Over-Provisioned

For a parameter $\beta \in (0,1)$,call a selfish routing network with $M/M/1$ delay functions β-Over-Provisioned if $f_e \leq (1-\beta)u_e$ for every edge e, where f is an equilibrium flow

At Equilibrium

The maximum link utilization in the network is at most $(1 - \beta)100\%$

POA Bounds for Over-Provisioned Networks



(a) M/M/1 delay function

(b) Extra capacity vs. POA curve

Figure 1: Modest overprovisioning guarantees near-optimal routing. The left-hand figure displays the per-unit cost $c(x) = 1/(u - x)$ as a function of the load x for an edge with capacity $u = 2$. The right-hand figure shows the worst-case price of anarchy as a function of the fraction of unused network capacity.

Intuition Suggested by Figure 1(a)

- When β is not too close to 0, the equilibrium flow is not too close to the capacity on any edge
- In this range the edges' cost functions behave like low-degree polynomials with nonnegative coefficients

From Last lecture

The POA is small in networks with such cost functions

Tight POA Bounds for Selfish Routing

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Theorem 5.1 (Tight POA Bounds for Selfish Routing (Informal)

Among all networks with cost functions in a set C, the largest POA is achieved in a Pigou-like network.

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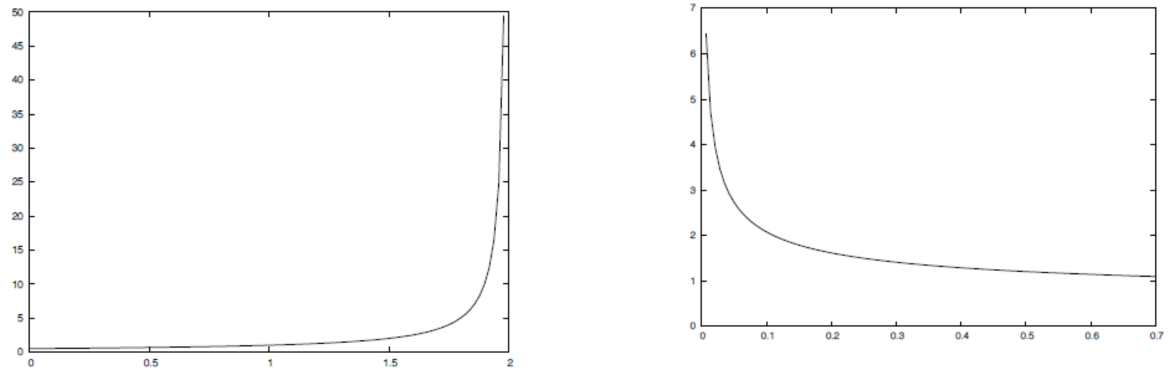
Theorem 5.2 (Tight POA Bounds for Selfish Routing (Formal)

For every set C of cost functions and every selfish routing network with cost functions in C, the POA is at most $\alpha(C)$.

POA Bounds for Over-Provisioned Networks



(a) M/M/1 delay function (b) Extra capacity vs. POA curve

Figure 1: Modest overprovisioning guarantees near-optimal routing. The left-hand figure displays the per-unit cost $c(x) = 1/(u - x)$ as a function of the load x for an edge with capacity $u = 2$. The right-hand figure shows the worst-case price of anarchy as a function of the fraction of unused network capacity.

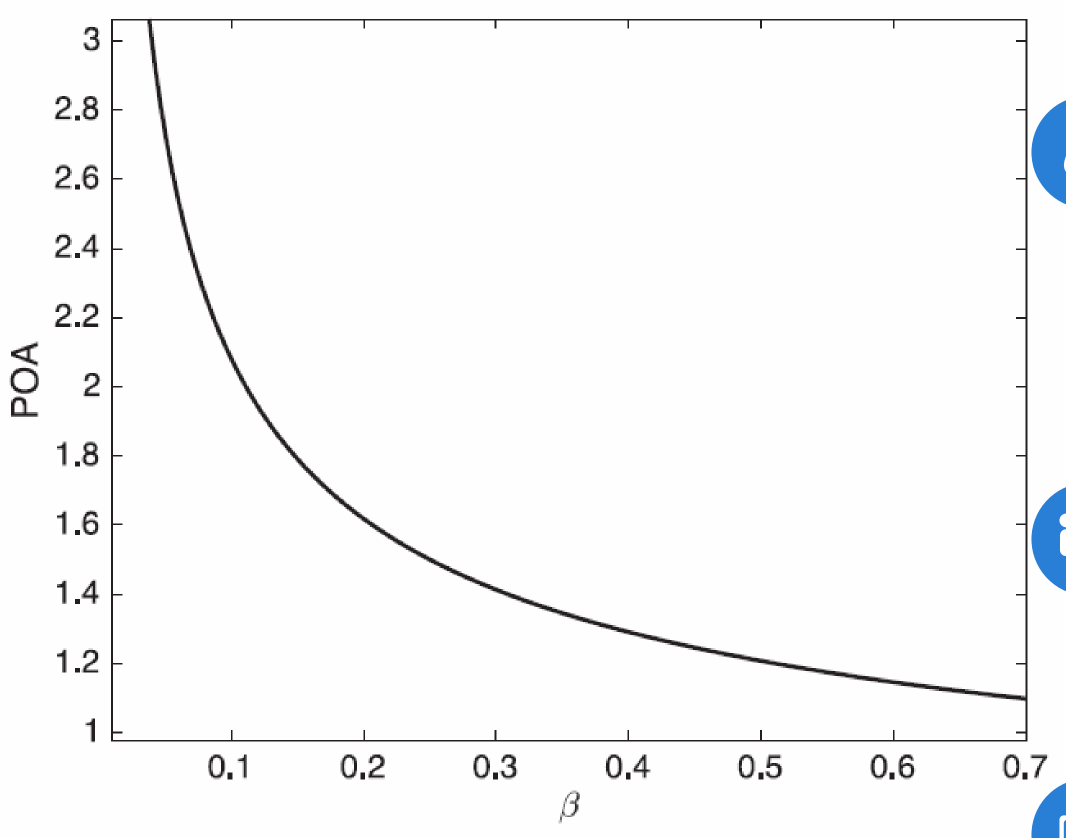
POA in β-Over-Provisioned

A computation shows that the worst-case POA in β-Over-Provisioned networks is at most

$$\frac{1}{2}\left(1 + \sqrt{\frac{1}{\beta}}\right)$$

an expression graphed in Figure 1(b)

POA Bounds for Over-Provisioned Networks



(b) Extra capacity vs. POA curve

POA Bound in (2)

The bound in (2) tends to 1 as β tends to 1 and to $+\infty$ as β tends to 0

- Where the cost functions effectively act like constant functions and like very high-degree polynomials, respectively

Intermediate Values of β

If β= 0.1 — meaning the maximum edge utilization is at most 90% — then the POA is guaranteed to be at most 2.1

Benefit of Over-Provisioning

A little over-provisioning is sufficient for near-optimal selfish routing, corroborating what has been empirically observed by Internet Service Providers

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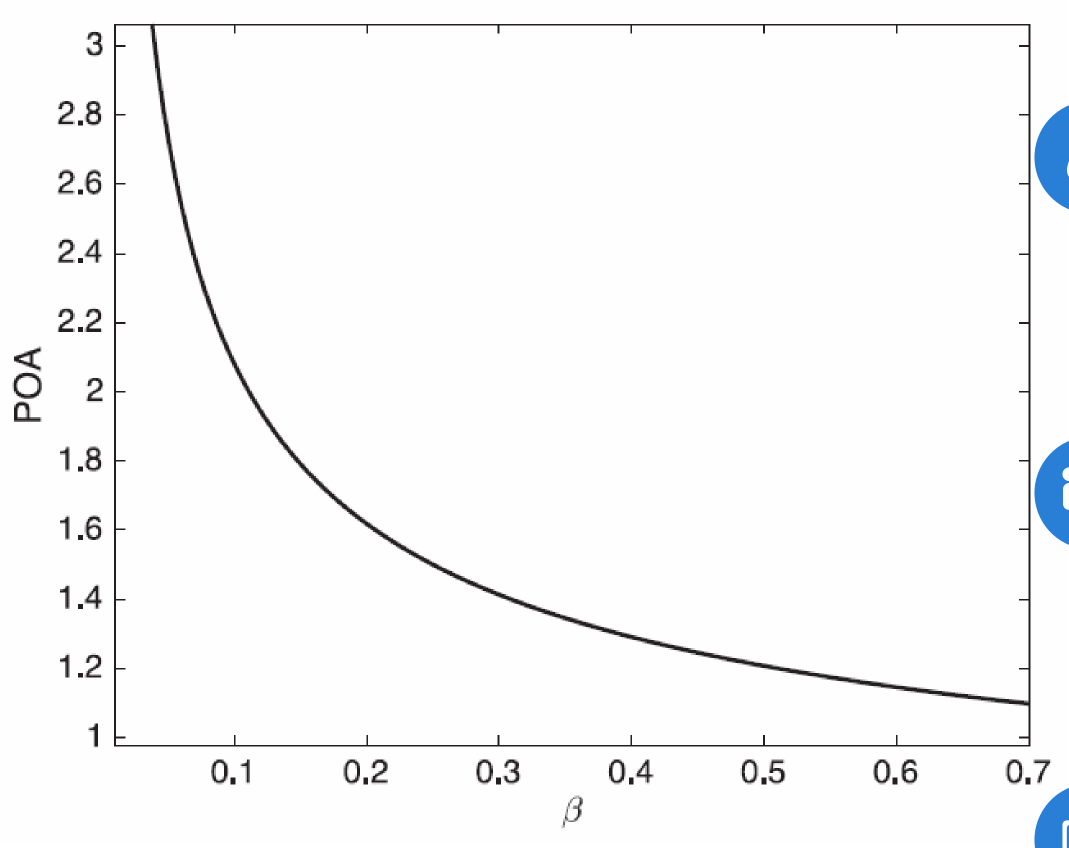
A Resource Augmentation Bound



(b) Extra capacity vs. POA curve

Section Goal

Proves a guarantee for selfish routing in arbitrary networks, with no extra assumptions on the cost function

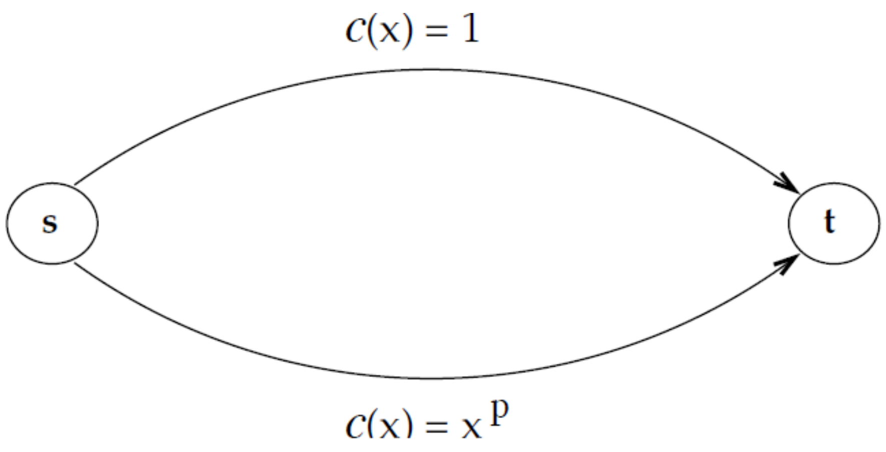
What Could Such a Guarantee Look Like?

What can we learn from the previous examples?

Nonlinear Variant of Pigou's Example

The POA of selfish routing can be arbitrarily large

A Resource Augmentation Bound



The Key Idea

Compare the performance of selfish routing to a handicapped minimum-cost solution that is forced to route extra traffic

With One Unit of Traffic

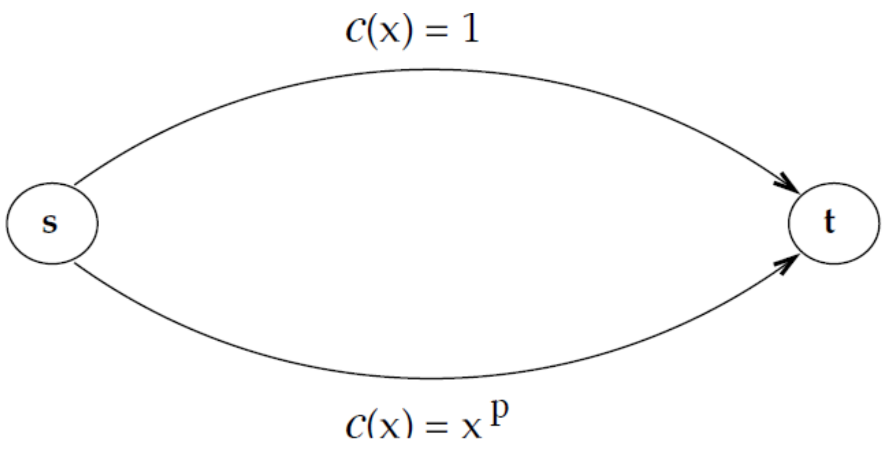
With one unit of traffic, the equilibrium flow has cost 1 while the optimal flow has near-zero cost

With Two Unit of Traffic

In the optimal flow, route two units of traffic through the network:

- The best solution continues to route $(1-\epsilon)$ units of traffic on the upper edge
- With the remaining $(1 + \epsilon)$ units of traffic routed on the lower edge
- For a total cost exceeding that of the equilibrium flow (with one unit of traffic)

A Resource Augmentation Bound



"Unfair" Comparison

There is an equivalent and easier to interpret formulation

- As a comparison between two flows with the same traffic rate but in networks with different cost functions

A "Faster" Network

Instead of forcing the optimal flow to route additional traffic

- Allow the equilibrium flow to use a "faster" network, with each original cost function $c_e(x)$ replaced by the "faster" function $c_e(x/2)/2$

A Resource Augmentation Bound

A "Faster" Network

Instead of forcing the optimal flow to route additional traffic

- Allow the equilibrium flow to use a "faster" network, with each original cost function $c_e(x)$ replaced by the "faster" function $c_e(x/2)/2$

The Transformation

This transformation is particularly easy to interpret for M/M/1 delay functions, since if $c_e(x) = 1/(u_e - x)$, then the "faster" function is $1/(2u_e - x)$— an edge with double the capacity

Insights

After this reformulation, gives a second justification for network over-provisioning

A modest technology upgrade improves performance more than implementing dictatorial control

A Resource Augmentation Bound

1

Theorem 6.1 (Resource Augmentation Bound)

For every selfish routing network and traffic rate r, the cost of an equilibrium flow with rate r is at most the cost of an optimal flow with rate $2r$

A modest technology upgrade improves performance more than implementing dictatorial control

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A Resource Augmentation Bound

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Theorem 6.1 (Resource Augmentation Bound)

For every selfish routing network and traffic rate r, the cost of an equilibrium flow with rate r is at most the cost of an optimal flow with rate $2r$

System Model

- Fix a network G with nonnegative, nondecreasing, and continuous cost functions, and a traffic rate r
- Let f and f^* denote equilibrium and optimal (minimum-cost) flows at the traffic rates r and $2r$, respectively

First Part of the Proof

- Reuses the trick from last lecture of using fictitious cost functions, frozen at the equilibrium costs, to get a grip on the cost of the optimal flow f^*
- Recall that since f is an equilibrium flow, all paths P used by f have a common cost $c_P(f)$, call it L

A Resource Augmentation Bound

1

Theorem 6.1 (Resource Augmentation Bound)

For every selfish routing network and traffic rate r, the cost of an equilibrium flow with rate r is at most the cost of an optimal flow with rate 2r.

First part of the proof

Moreover, $c_P(f) \geq L$ for every path $P \in \mathcal{P}$, we have

$$\sum_{e \in E} f_e \cdot c_e(f_e) = \sum_{P \in \mathcal{P}} f_P \cdot c_p(f) = r \cdot L$$

$$\sum_{e \in E} f_e^* \cdot c_e(f_e) = \sum_{P \in \mathcal{P}} f_P^* \cdot c_p(f) \geq 2r \cdot L$$

With respect to the fictitious costs $c_e(f_e)$, we get a great lower bound on the cost of f^* — at least twice the cost of the equilibrium flow f — much better than what we're actually trying to prove

A Resource Augmentation Bound

1

Theorem 6.1 (Resource Augmentation Bound)

For every selfish routing network and traffic rate r, the cost of an equilibrium flow with rate r is at most the cost of an optimal flow with rate 2r.

Second Part of the Proof

Shows that using the fictitious costs instead of the accurate ones overestimates the cost of f^* by at most the cost of f. Specifically, we complete the proof by showing that

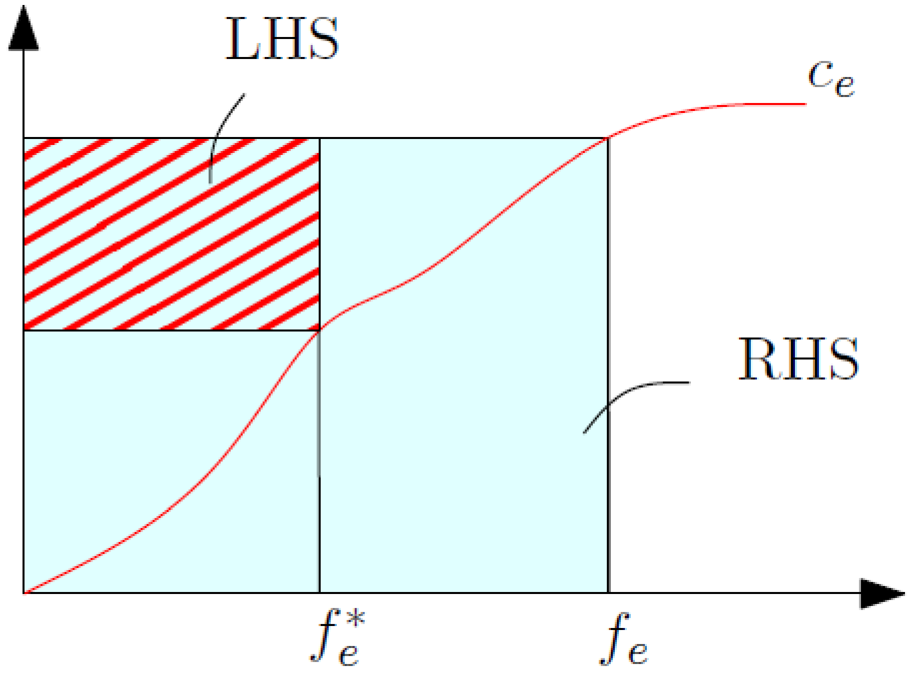
$$\sum_{e \in E} f_e^* \cdot c_e(f_e^*) \geq \sum_{e \in E} f_e^* \cdot c_e(f_e) - \sum_{e \in E} f_e \cdot c_e(f_e)$$

Prove That it Holds Term-by-Term

$$\sum_{e \in E} f_e^* \cdot [c_e(f_e) - c_e(f_e^*)] \leq \sum_{e \in E} f_e \cdot c_e(f_e)$$

For every edge $e \in E$. When $f_e^* \geq f_e$, the left-hand side is non-positive and there is nothing to show

A Resource Augmentation Bound



When $f_e^* < f_e$

A proof by picture

$$\sum_{e \in E} f_e^* \cdot [c_e(f_e) - c_e(f_e^*)] \leq \sum_{e \in E} f_e \cdot c_e(f_e)$$

The Left-Hand Side

The area of the shaded region, with width f_e^* and height $c_e(f_e) - c_e(f_e^*)$

The Right-Hand Side

The area of the solid region, with width f_e and height $c_e(f_e)$

Complete the Proof

Since $f_e^* < f_e$ and c_e is nondecreasing, the former region is a subset of the latter

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OUTLINE

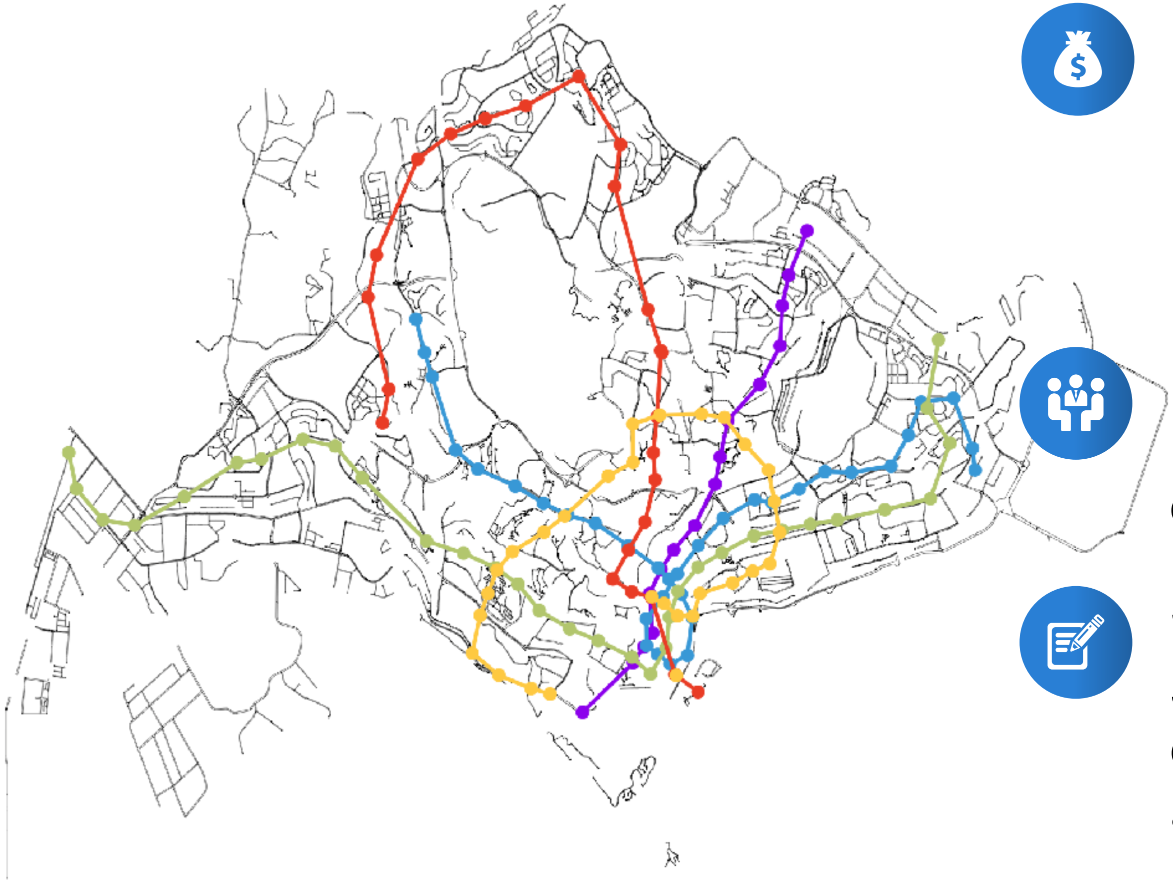
01 Case Study: Network Over-Provisioning

02 A Resource Augmentation Bound

03 Proof of Theorem 6.1

04 Atomic Selfish Routing

Atomic Selfish Routing



So far

- Non-atomic model of selfish routing, meaning that all players were assumed to have negligible size
- A good model for cars on a highway or small users of a communication network

However

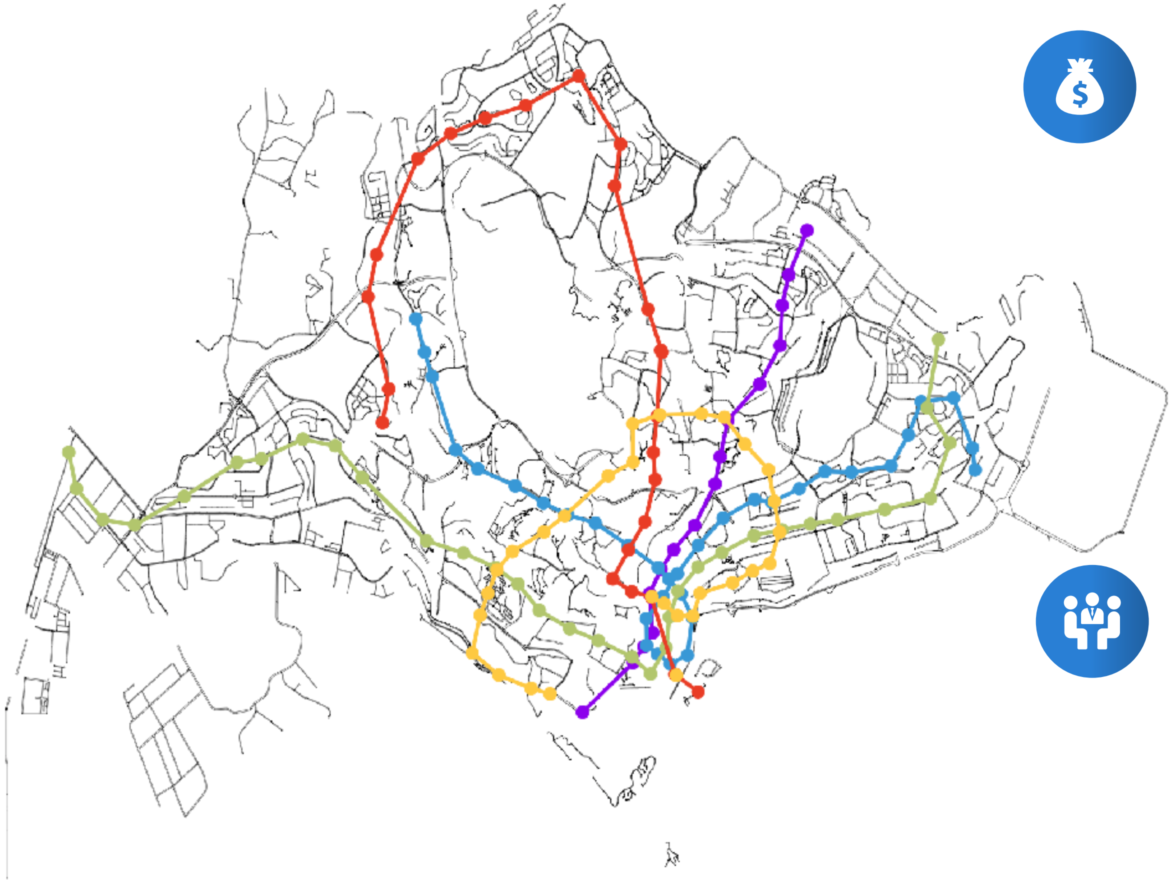
But not if a single strategic player represents, for example, all of the traffic controlled by a single Internet Service Provider

Section Goal

Studies atomic selfish routing networks, where each player controls a non-negligible amount of traffic

- While most aspects of the model will be familiar, it presents a couple of new technical complications

Atomic Selfish Routing



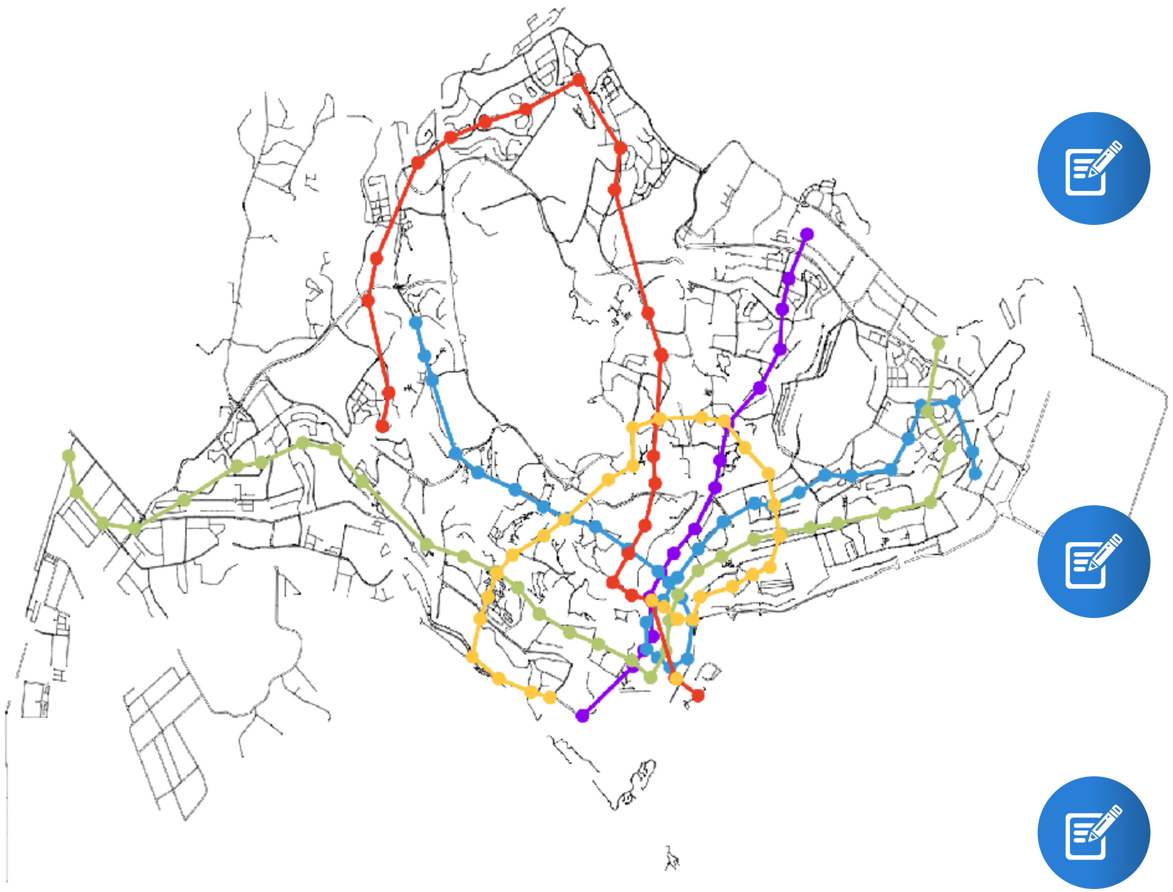
System Model

- An atomic selfish routing network consists of a directed graph $G = (V, E)$
- With nonnegative and non-decreasing edge cost functions
- A finite number k of agents

Agent Model

- Agent i has a origin vertex o_i and a destination vertex d_i
- Each agent routes 1 unit of traffic on a single $o_i - d_i$ path, and seeks to minimize her cost
- These can be shared across agents, or not

Atomic Selfish Routing



Path and Flows

- Let P_i denote the $o_i - d_i$ paths of G
- A flow can now be represented as a vector $(P_1, \ldots, P_k)$, with $P_i \in \mathcal{P}_i$ the path on which agent i routes her traffic

Cost

The cost of a flow is defined as in the non-atomic model

Equilibrium Flow

An equilibrium flow is one in which no agent can decrease her cost via a unilateral deviation

Second-Price Auctions and Dominant Strategies

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Definition 6.2 (Equilibrium Flow (Atomic))

A flow $(P_1, \ldots, P_k)$ is an equilibrium if, for every agent i and path $P_i \in \mathcal{P}_i$

$$\sum_{e \in P_i} c_e(f_e) \leq \sum_{e \in \hat{P}_i \cap P_i} c_e(f_e) + \sum_{e \in \hat{P}_i \setminus P_i} c_e(f_e + 1)$$

Comparison

Definition 6.2 differs from Definition 6.3 because a deviation by an agent with non-negligible size increases the cost of the newly used edges

Atomic Selfish Routing

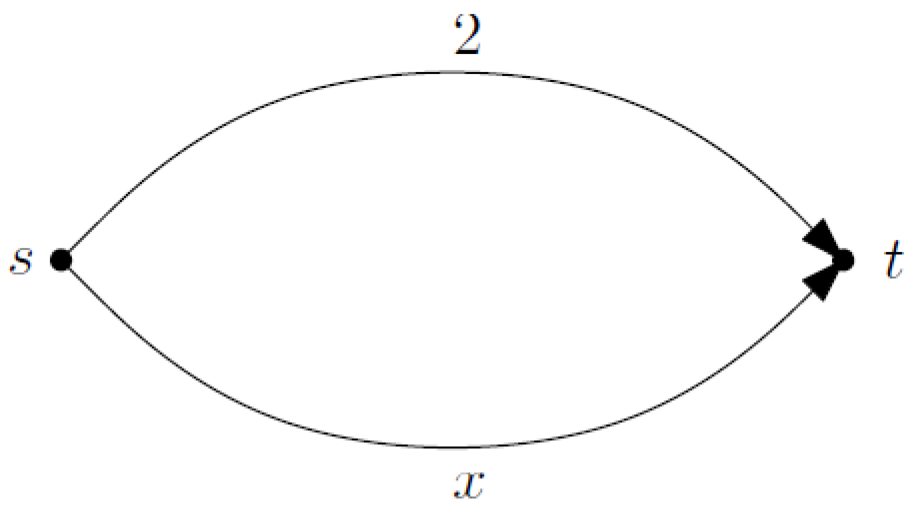

Figure 4: A pigou-like network for atomic selfish routing.

Variant of Pigou’s Example

To get a feel for the atomic model, consider the variant of Pigou’s example shown in Figure 4

Flow and Cost

- Suppose there are two players, and recall that each controls 1 unit of flow
- The optimal solution routes one player on each link, for a total cost of 1 + 2 = 3

Atomic Selfish Routing

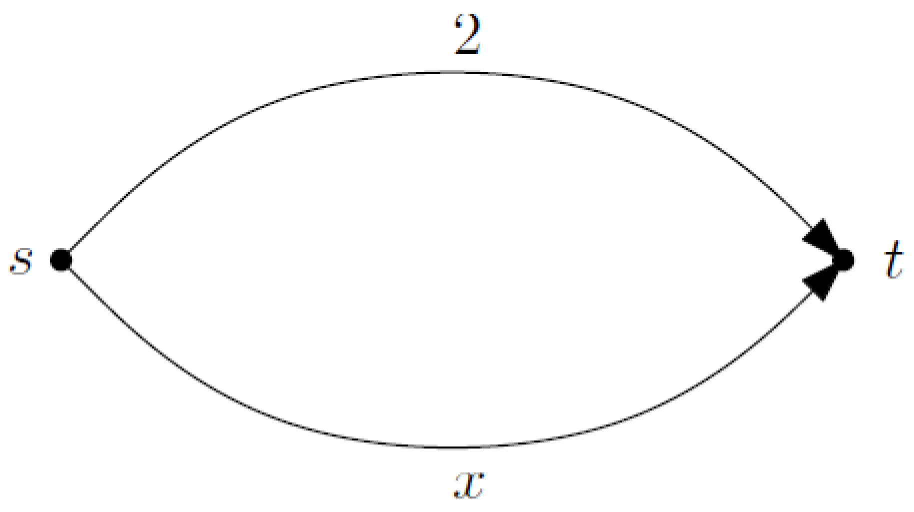

Figure 4: A Pigou-like network for atomic selfish routing.

Player on the Lower Edge

The player on the lower edge does not want to switch, since its cost would jump from 1 to 2

Player on the Upper Edge

With cost 2, has no incentive to switch to the bottom edge, where its sudden appearance would drive the cost up to 2

Equilibrium Flow

This is also an equilibrium flow, in the sense that neither player can decrease its cost via a unilateral deviation

Atomic Selfish Routing

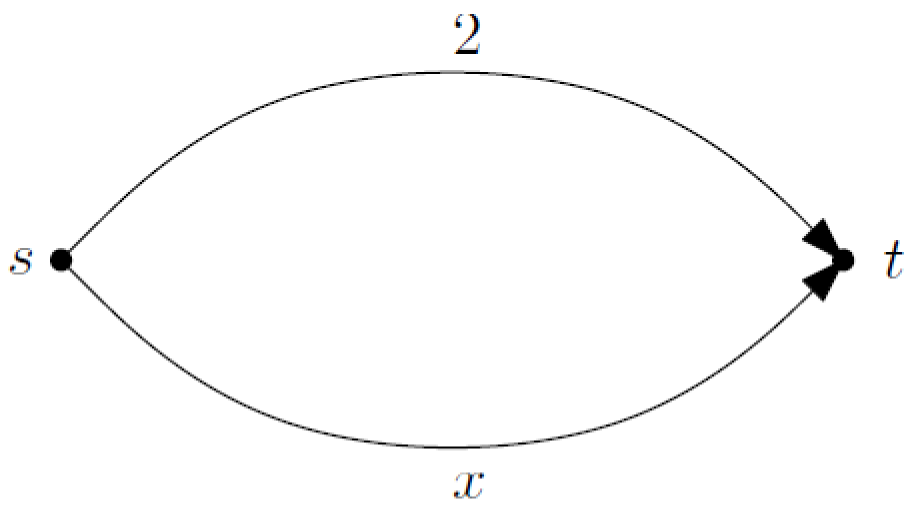

Figure 4: A pigou-like network for atomic selfish routing.

A Second Equilibrium

If both players take the lower edge, both have a cost of 2 and neither can decrease its cost by switching to the upper edge

Cost

This equilibrium has cost 4

Difference

This illustrates an importance difference between the non-atomic and atomic models

- Different equilibria are guaranteed to have the same cost in the non-atomic model, but not in the atomic model

Second-Price Auctions and Dominant Strategies

2 Current Definition of the POA

The ratio between the objective function value of an equilibrium and that of an optimal outcome

Disadvantage of the Current Definition

Our current working definition of the POA is not well defined when different equilibria have different objective function values

Extend the Definition

We extend the definition by taking a worst-case approach: the price of anarchy (POA) of an atomic selfish routing network is

$$\frac{cost\ of\ worst\ equilibrium}{cost\ of\ optimal\ outcome}$$

For Example

In the network in Figure 4, the POA is 4/3

Atomic Selfish Routing

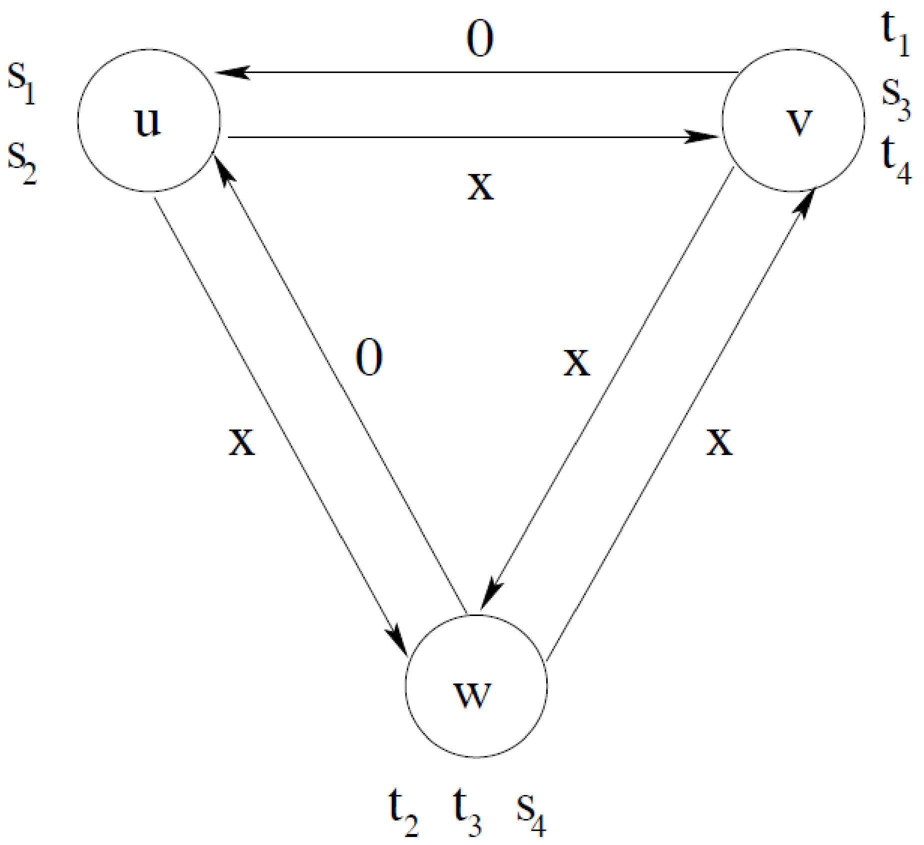

Figure 5: In atomic instances with affine cost functions, the POA can be as large as 5/2.

A Second Difference Between Two Models

POA in atomic selfish routing networks can be larger than in their non-atomic counterparts

To See This

Consider the four-player bidirected triangle network shown in Figure 5

Two Strategies

Each player has two strategies, a one-hop path and a two-hop path

Atomic Selfish Routing

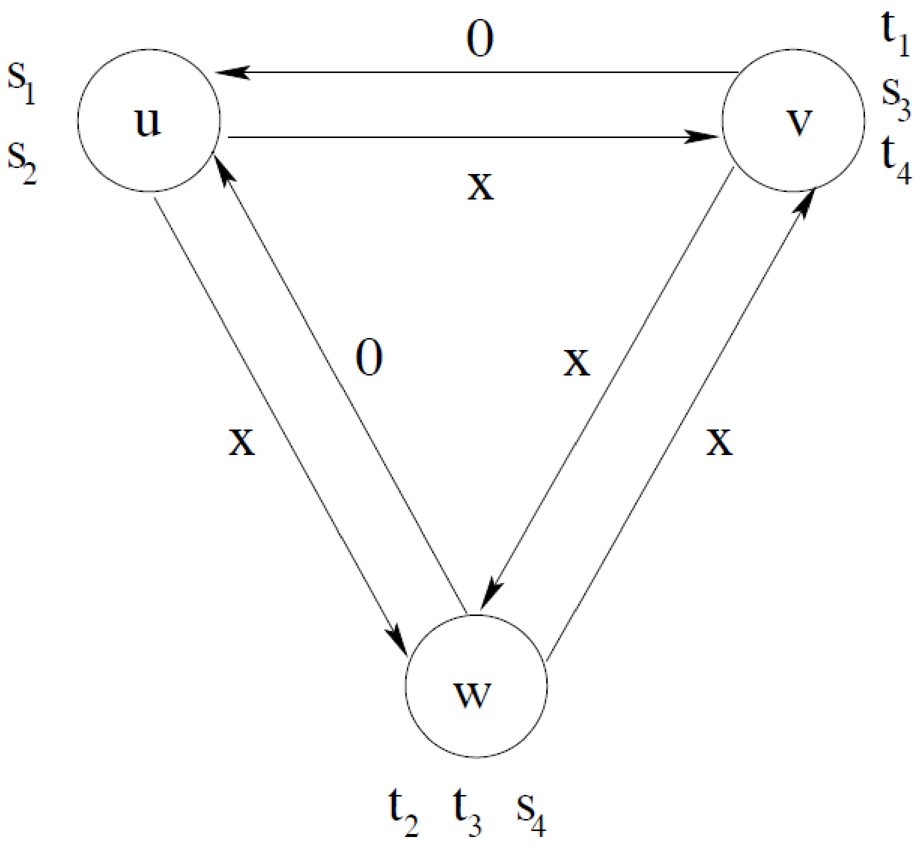

Figure 5: In atomic instances with affine cost functions, the POA can be as large as 5/2.

One-Hop Path Equilibrium

All players route on their one-hop paths, and the cost of this flow is 4

- These one-hop paths are precisely the four edges with the cost function $c(x) = x$

Two-Hop Paths Equilibrium

The first two players each incur three units of cost and the last two players each incur two units of cost, this flow has a cost of 10

- The price of anarchy of this instance is therefore 10/4 = 2.5

Any Other Case?

There are no worse examples with affine cost functions

Second-Price Auctions and Dominant Strategies

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Theorem 6.3
(POA Bound for Atomic Selfish Routing, Affine Cost Function)

In every atomic selfish routing network with affine cost functions, the POA is at most 5/2

Tight POA Bounds

Theorem 6.3 and its proof can be generalized to give tight POA bounds for arbitrary sets of cost functions

Proof of Theorem 6.3

The proof of Theorem 12.3 is a “canonical POA proof”

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Theorem 6.3 (POA Bound for Atomic Selfish Routing)

In every atomic selfish routing network with affine cost functions, the POA is at most 5/2

Define the Functions

- Bound from above the cost of every equilibrium flow; fix one f arbitrarily
- Let f^* denote a minimum-cost flow
- Write f_e and f_e^* for the number of agents in f and f^*, respectively, that pick a path that includes the edge e

The First Step of the roof

Identifies a useful way of applying our hypothesis that f is an equilibrium flow

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Theorem 6.3 (POA Bound for Atomic Selfish Routing)

In every atomic selfish routing network with affine cost functions, the POA is at most 5/2

Consider the Case

If any agent i, using the path P_i in f, and any unilateral deviation to a different path $\hat{P}_i$, then we can conclude that i's equilibrium cost using P_i is at most what her cost would be if she switched to $\hat{P}_i$ (Definition 6.2)

Then..

We want an upper bound on the cost of the equilibrium flow f, and hypothetical deviations give us upper bounds on the equilibrium costs of individual agents

Second-Price Auctions and Dominant Strategies

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Theorem 6.3 (POA Bound for Atomic Selfish Routing)

In every atomic selfish routing network with affine cost functions, the POA is at most 5/2

Which hypothetical deviations should we single out for the proof?

Given that f^* is the only other object referenced in the theorem statement, a natural idea is to use the optimal flow f^* to suggest deviations

Formally

Suppose agent i using the path P_i in f, and P_i^* in f^*, by Definition 6.2

$$\sum_{e \in P_i} c_e(f_e) \leq \sum_{e \in P_i^* \cap P_i} c_e(f_e) + \sum_{e \in P_i^* \setminus P_i} c_e(f_e + 1)$$

This completes the first step, in which we apply the equilibrium hypothesis to generate an upper bound (12.5) on the equilibrium cost of each agent

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Theorem 6.3 (POA Bound for Atomic Selfish Routing)

In every atomic selfish routing network with affine cost functions, the POA is at most 5/2

The Second Step of the Proof

Sums the upper bound (12.5) on individual equilibrium costs over all agents to obtain a bound on the total equilibrium cost

$$\underbrace{\sum_{i=1}^{k}\sum_{e\in P_i} c_e(f_e)}_{\text{cost of } f} \le \sum_{i=1}^{k}\left(\sum_{e\in P_i^*\cap P_i} c_e(f_e) + \sum_{e\in P_i^*\setminus P_i} c_e(f_e+1)\right)$$

$$\le \sum_{i=1}^{k}\sum_{e\in P_i^*} c_e(f_e+1)$$

$$= \sum_{e\in E} f_e^* \cdot c_e(f_e+1)$$

$$= \sum_{e\in E} \left[a_e f_e^*(f_e+1) + b_e f_e^*\right],$$

6

Theorem 6.3 (POA Bound for Atomic Selfish Routing)

In every atomic selfish routing network with affine cost functions, the POA is at most 5/2

The Inequality

Cost functions are non-decreasing

The Equation

The term $c_e(f_e + 1)$is contributed once by each agent i for which $e \in P_i^*$ (f_e^* times in all), and equation (12.9) from the assumption that cost functions are affine

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Theorem 6.3 (POA Bound for Atomic Selfish Routing)

In every atomic selfish routing network with affine cost functions, the POA is at most 5/2

Upper Bound

The previous step gives an upper bound on a quantity that we care about—the cost of the equilibrium flow f—in terms of a quantity that we don't care about, the "entangled" version of f and f^* on the right-hand side

Third Step of the Proof

The third and most technically challenging step of the proof is to "disentangle" the right-hand side and relate it to the only quantities that we care about for a POA bound, the costs of f and f^*

Second-Price Auctions and Dominant Strategies

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Lemma 6.4

For every $y, z \in \{0, 1, 2, 3, \ldots\}$,

$$y(z+1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2$$

Upper Bound

The previous step gives an upper bound on a quantity that we care about—the cost of the equilibrium flow f—in terms of a quantity that we don't care about, the "entangled" version of f and f^* on the right-hand side

Second-Price Auctions and Dominant Strategies

6

Lemma 6.4

For every $y, z \in \{0, 1, 2, 3, \ldots\}$,

$$y(z+1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2$$

Third Step of the Proof

We now apply Lemma 6.4 once per edge in the right-hand side, with $y = f_e^*$ and $z = f_e$. Using the definition of the cost $C(\cdot)$ of a flow, this yields

$$\begin{aligned} C(f) &\leq \sum_{e\in E}\left[a_e\left(\frac{5}{3}(f_e^*)^2 + \frac{1}{3}f_e^2\right) + b_e f_e^*\right] \\ &\leq \frac{5}{3}\left[\sum_{e\in E} f_e^*(a_e f_e^* + b_e)\right] + \frac{1}{3}\sum_{e\in E} a_e f_e^2 \\ &\leq \frac{5}{3}\cdot C(f^*) + \frac{1}{3}\cdot C(f). \end{aligned}$$

Second-Price Auctions and Dominant Strategies

6 Theorem 6.3 (POA Bound for Atomic Selfish Routing)

In every atomic selfish routing network with affine cost functions, the POA is at most 5/2

$$\begin{aligned} C(f) &\le \sum_{e\in E}\left[a_e\left(\frac{5}{3}(f_e^*)^2+\frac{1}{3}f_e^2\right)+b_e f_e^*\right] \\ &\le \frac{5}{3}\left[\sum_{e\in E} f_e^*(a_e f_e^* + b_e)\right]+\frac{1}{3}\sum_{e\in E} a_e f_e^2 \\ &\le \frac{5}{3}\cdot C(f^*)+\frac{1}{3}\cdot C(f). \end{aligned}$$

second step of the proof

Subtracting $1/3C(f)$ from both sides and multiplying through by 3/2 gives

$$C(f) \le \frac{5}{3}\cdot\frac{3}{2}\cdot C(f^*) = \frac{5}{2}\cdot C(f^*)$$

which completes the proof of Theorem 6.3

The Upshots

β-Over-Provisioned

A selfish routing network with cost functions of the form $c_e(x) = 1/(u_e - x)$ is β-Over-Provisioned if the amount of equilibrium flow on each edge e is at most $(1 - \beta)u_e$

POA of β-Over-Provisioned Networks

The POA is small in β-over-provisioned networks even with fairly small β, corroborating empirical observations that a little over-provisioning yields good network performance

Cost of an Equilibrium Flow VS Optimal Flow

- The cost of an equilibrium flow is at most that of an optimal flow that routes twice as much traffic
- Equivalently, a modest technology upgrade improves performance more than implementing dictatorial control

The Upshots

Different Equilibria

In atomic selfish routing, where each agent controls a non-negligible fraction of the network traffic, different equilibrium flows can have different costs

New Definition of POA

The POA is the ratio between the objective function value of the worst equilibrium and that of an optimal outcome

Worst-Case POA of Atomic Selfish Routing

The worst-case POA of atomic selfish routing with affine cost functions is exactly 2.5

李永乐老师

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