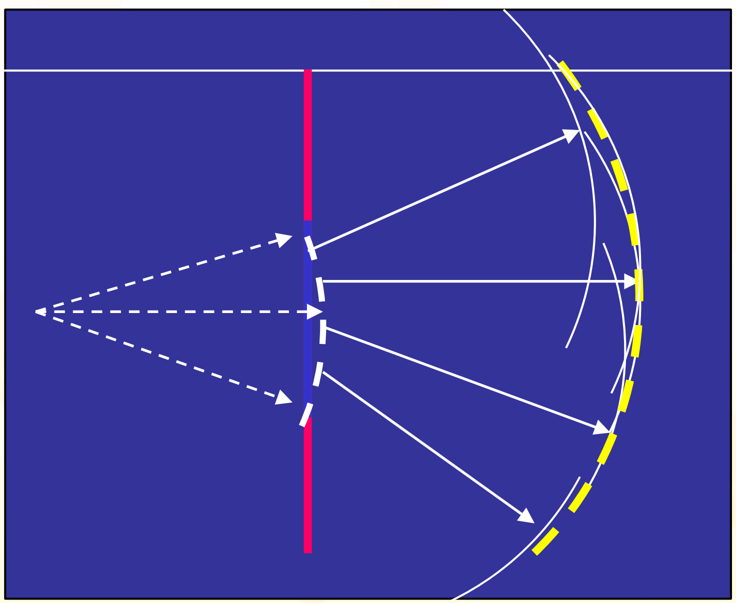
# 第六章 积分方程方法

6.1 表面积分方程
6.2 体积分方程
6.3 散射问题的近似解

# Huygens-Fresnel原理

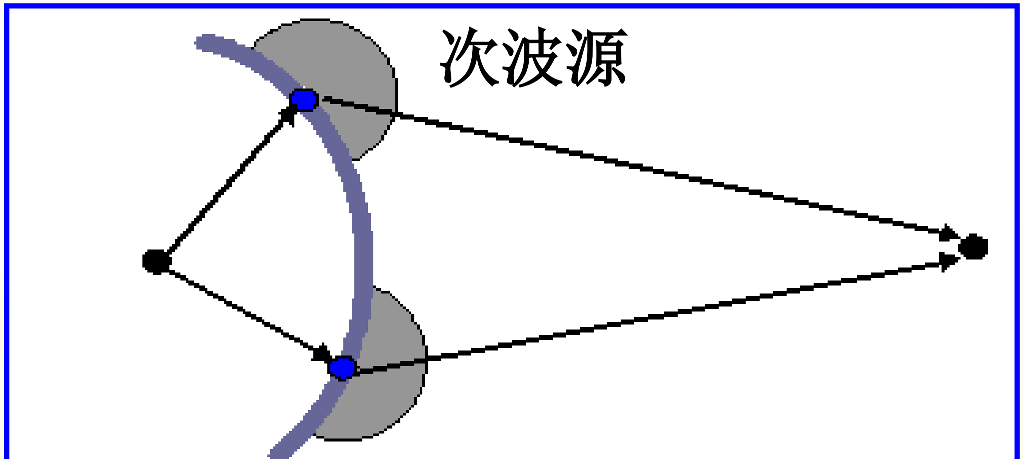
Huygens在研究波动现象时指出：波在传播过程中，波阵面上的每一点都是产生球面子波的次级波源，以后任意时刻的波阵面都是上一级子波的贡献之和。



惠更斯：后一级波阵面是上一级波阵面子波的贡献之和。

## Fresnel在研究Huygens原理的基础上认为：

二次辐射产生的球面子波是相互干涉的，空间任意点的场是波阵面上的所有次级波源发出的球面波在该点的干涉叠加，进一步完善了Huygens原理，称为Huygens-Fresnel原理。

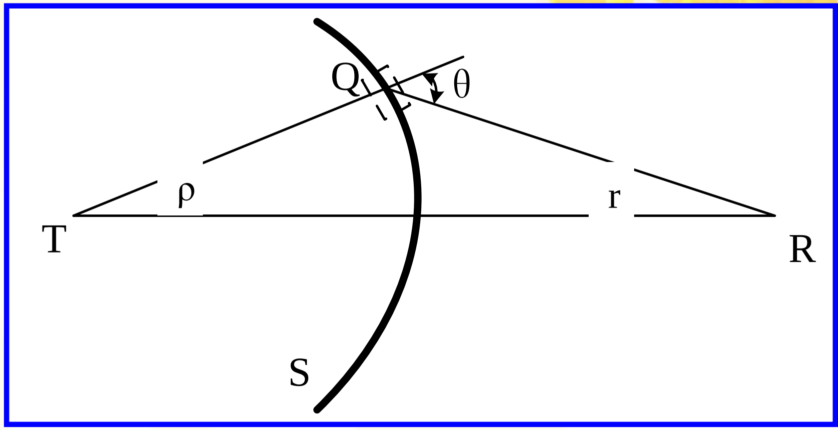



Huygens-Fresnel 原理

**惠更斯面：包围波源的任一封闭面**

**菲涅尔：空间任一点的场等于惠更斯面上各次级波源在该点所产生场的叠加。**

# 惠更斯-菲涅尔原理数学表达式:




$$E_Q = A\frac{\exp(-jk\rho)}{\rho}$$

$$dE = ds\cdot\cos\theta\cdot A\frac{\exp(-jk\rho)}{\rho}\cdot A'\frac{\exp(-jkr)}{r}$$

$$E = AA'\frac{\exp(-jk\rho)}{\rho}\iint_s\frac{\exp(-jkr)}{r}\cdot\cos\theta\cdot ds$$

$$\boxed{A' = -\frac{j}{\lambda}}$$

$$E = -\frac{jA}{\lambda}\iint_s\frac{\exp[-jk(r+\rho)]}{r\cdot\rho}\cdot\cos\theta\cdot ds$$

$$E = -\frac{j}{\lambda}\iint_s E(r')\frac{\exp(-jkr)}{r}\cdot\cos\theta\cdot ds$$

# 8.1 表面积分方程

## 8.1.1 标量波方程

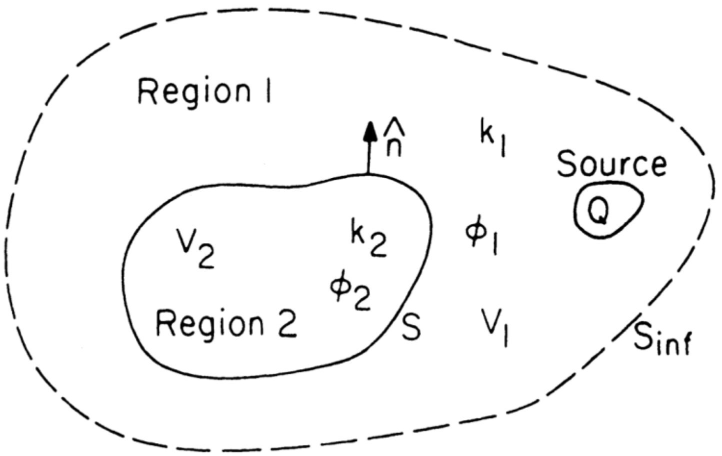



**Figure 8.1.1** A two-region problem can be solved with a surface integral equation.

区域1：

$$(\nabla^2 + k_1^2)\,\phi_1(\mathbf{r}) = Q(\mathbf{r}), \tag{8.1.1}$$

$$(\nabla^2 + k_1^2)\,g_1(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'), \tag{8.1.3}$$

$$\int_{V_1} dV\,[g_1(\mathbf{r},\mathbf{r}')\nabla^2\phi_1(\mathbf{r}) - \phi_1(\mathbf{r})\nabla^2 g_1(\mathbf{r},\mathbf{r}')]$$
$$= \int_{V_1} dV\, g_1(\mathbf{r},\mathbf{r}')Q(\mathbf{r}) + \phi_1(\mathbf{r}'), \quad \mathbf{r}' \in V_1. \quad (8.1.5)$$

$$-\int_{S+S_{inf}} dS\,\hat{n}\cdot[g_1(\mathbf{r},\mathbf{r}')\nabla\phi_1(\mathbf{r}) - \phi_1(\mathbf{r})\nabla g_1(\mathbf{r},\mathbf{r}')]$$
$$= -\phi_{inc}(\mathbf{r}') + \phi_1(\mathbf{r}'), \quad \mathbf{r}' \in V_1. \quad (8.1.6)$$

$$\phi_{inc}(\mathbf{r}') = -\int_{V_1} dV\, g_1(\mathbf{r},\mathbf{r}')Q(\mathbf{r}), \quad (8.1.7)$$

无界空间格林函数：

$$g_1(\mathbf{r},\mathbf{r}') = \frac{e^{ik_1|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \quad (8.1.8)$$

无源空间标量波函数解为：

$$\varphi(\mathbf{r})=\oiint_{-S+S_{\text{inf}}}\left[G(\mathbf{r},\mathbf{r}')\nabla'\varphi(\mathbf{r}')-\varphi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')\right]\cdot d\mathbf{S}'$$

标量场吉尔霍夫(Kirchhoff)公式

$$\nabla'G(\mathbf{r},\mathbf{r}')=\hat{R}(jk+\frac{1}{R})\frac{\exp(-jkR)}{4\pi R}=\hat{R}(jk+\frac{1}{R})G(\mathbf{r},\mathbf{r}')$$

球面波幅度因子

$$\varphi(\mathbf{r})=\frac{1}{4\pi}\oiint_S\left[\nabla'\varphi(\mathbf{r}')-\hat{\mathbf{R}}\left(jk+\frac{1}{R}\right)\varphi(\mathbf{r}')\right]\frac{e^{-jkR}}{R}\cdot ds'$$

积分表示界面
所有次波叠加

球面波因子，表示发自
边界面上r’点的球面波

这正是Huygens-Fresnel原理的数学表达式。

表示区域内任意点r场是界面上所有次波源发出次波在该点干涉叠加的结果。

如果 $R \to \infty$ $\mathrm{d}\mathbf{s}' = \hat{R}R^2 \mathrm{d}\Omega$ $\nabla' \phi(\mathbf{r}') = \hat{R}\dfrac{\partial \phi}{\partial R}$

$$\varphi(\mathbf{r}) = \lim_{R\to\infty} \frac{1}{4\pi} \oiint_S R\left( \frac{\partial \varphi(\mathbf{r}')}{\partial R} - \mathrm{j}k\varphi(\mathbf{r}') \right) \mathrm{d}\Omega\, e^{-\mathrm{j}kR} - \lim_{R\to\infty} \frac{1}{4\pi} \oiint_S \varphi(\mathbf{r}') \mathrm{d}\Omega\, e^{-\mathrm{j}kR}$$

表示无穷远边界上次波源在空间内r点辐射场的叠加，其结果必为零。否则有限区域内电磁场因与无穷远边界上电磁场有关而具有多值特性。即：

$$\lim_{r\to\infty} \phi(\boldsymbol{r}) = 0$$

$$\lim_{r\to\infty} r\left( \frac{\partial \varphi(\mathbf{r})}{\partial r} - \mathrm{j}k\varphi(\mathbf{r}) \right) = 0$$

称为Sommerfeld辐射条件

无界空间标量波函数解为：

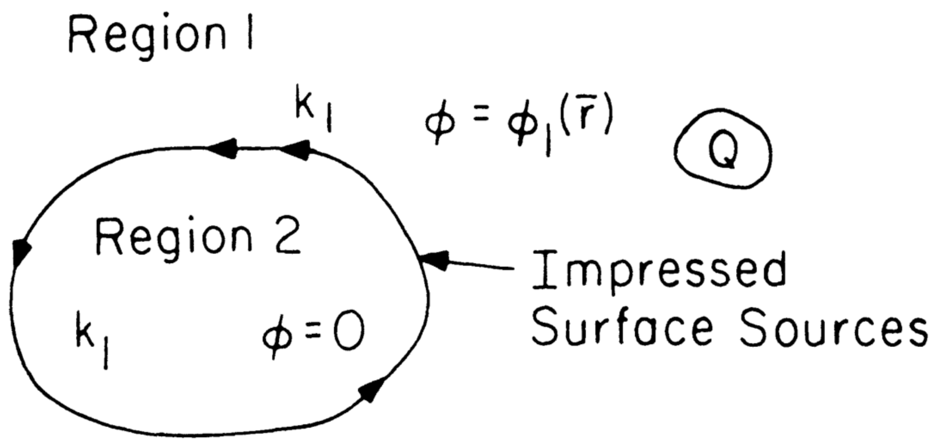



**Figure 8.1.2** The illustration of the extinction theorem.

$$\phi_1(\mathbf{r}) = \phi_{inc}(\mathbf{r}) - \int_S dS'\,\hat{n}'\cdot[g_1(\mathbf{r},\mathbf{r}')\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\nabla' g_1(\mathbf{r},\mathbf{r}')], \quad \mathbf{r}\in V_1. \tag{8.1.9}$$

若 $r \in V_2$ ？

$$\left.\begin{array}{ll}\mathbf{r}\in V_1, & \phi_1(\mathbf{r}) \\ \mathbf{r}\in V_2, & 0\end{array}\right\} = \phi_{inc}(\mathbf{r}) - \int_S dS'\,\hat{n}'\cdot[g_1(\mathbf{r},\mathbf{r}')\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\nabla' g_1(\mathbf{r},\mathbf{r}')]. \tag{8.1.10}$$

若S为不可穿透边界？

区域2：

$$(\nabla^2 + k_2^2)\,\phi_2(\mathbf{r}) = 0. \tag{8.1.2}$$

$$(\nabla^2 + k_2^2)\,g_2(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'). \tag{8.1.4}$$

由类似推导：

$$\left.\begin{array}{ll}\mathbf{r}\in V_2, & \phi_2(\mathbf{r})\\ \mathbf{r}\in V_1, & 0\end{array}\right\} = \int_S dS'\,\hat{n}'\cdot[g_2(\mathbf{r},\mathbf{r}')\nabla'\phi_2(\mathbf{r}') - \phi_2(\mathbf{r}')\nabla' g_2(\mathbf{r},\mathbf{r}')]. \tag{8.1.11}$$

应用自屏蔽定理，建立方程：

$$\left\{\begin{array}{l}\phi_{inc}(\mathbf{r}) = \int_S dS'\,\hat{n}'\cdot[g_1(\mathbf{r},\mathbf{r}')\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\nabla' g_1(\mathbf{r},\mathbf{r}')], \quad \mathbf{r}\in V_2, \qquad (8.1.12a)\\ 0 = \int_S dS'\,\hat{n}'\cdot[g_2(\mathbf{r},\mathbf{r}')\nabla'\phi_2(\mathbf{r}') - \phi_2(\mathbf{r}')\nabla' g_2(\mathbf{r},\mathbf{r}')], \quad \mathbf{r}\in V_1. \qquad (8.1.12b)\end{array}\right.$$

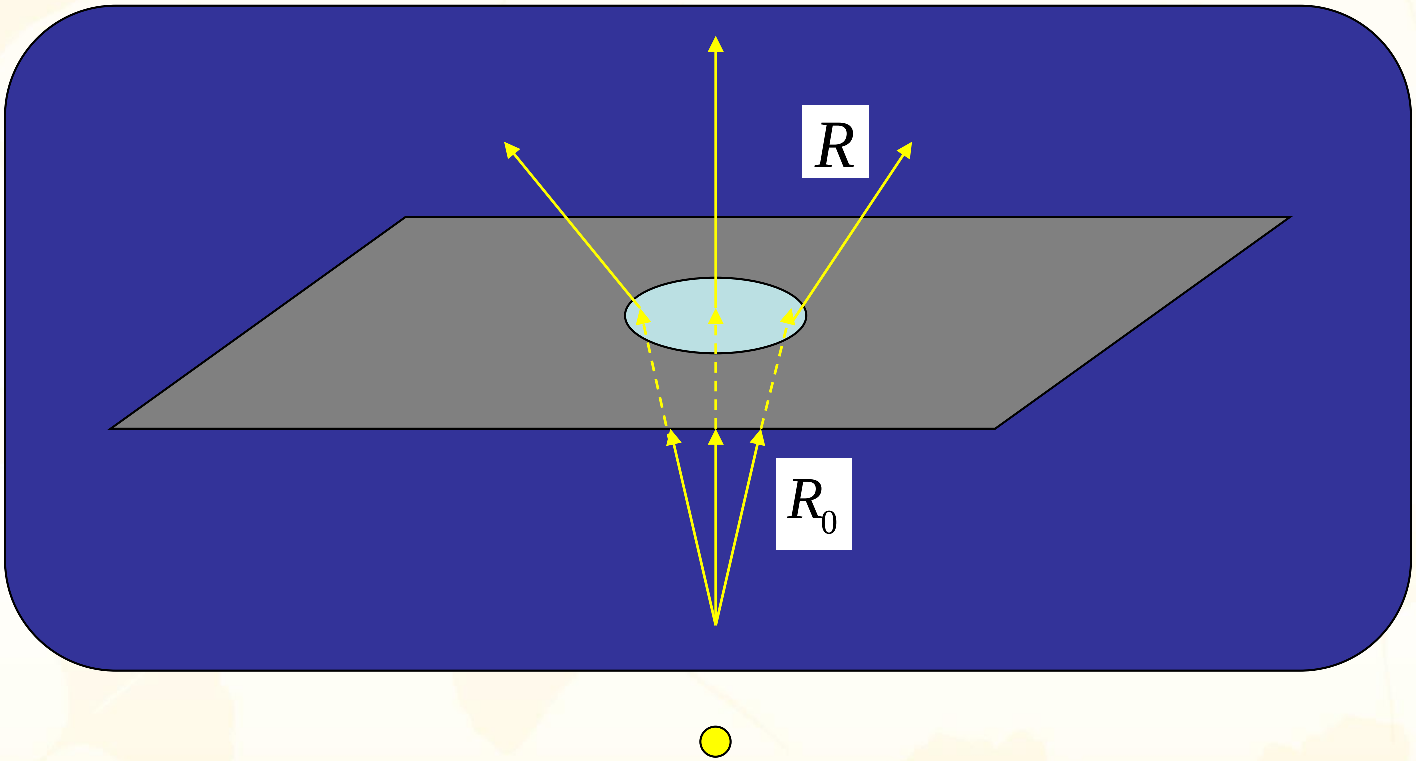
边界条件：

$$\left\{\begin{array}{ll}\phi_1(\mathbf{r}) = \phi_2(\mathbf{r}), \quad \mathbf{r}\in S, & (8.1.13a)\\ p_1\hat{n}\cdot\nabla\phi_1(\mathbf{r}) = p_2\hat{n}\cdot\nabla\phi_2(\mathbf{r}), \quad \mathbf{r}\in S. & (8.1.13b)\end{array}\right.$$

格林函数分别满足方程(3)，(4)即可，通常选择均匀介质格林函数。

例：小孔衍射

圆形小孔的半径为a，远大于波长




应用Kirchhoff公式，必须知道屏幕上 $\nabla'\phi(\mathbf{r}'),\phi(\mathbf{r}')$

假设：
（1）在小孔上，$\nabla'\phi(\boldsymbol{r}'),\phi(\boldsymbol{r}')$为点光源的直射场，即假设屏幕不对入射波产生影响。
（2）在小孔以外的屏幕上，$\nabla'\phi(\boldsymbol{r}'),\phi(\boldsymbol{r}')$ 恒为零。

在上述假设下，在屏幕小孔上

$$\phi(\boldsymbol{r}') = \frac{A}{R_0} exp(-\mathrm{j}kR_0)$$

$$\boldsymbol{R}_0 = \boldsymbol{r}' - \boldsymbol{r}_0$$

$$\nabla'\phi(\boldsymbol{r}') = -\left(\mathrm{j}k + \frac{1}{R_0}\right)\frac{1}{R_0}\exp(-\mathrm{j}kR_0)\hat{\boldsymbol{R}}_0$$

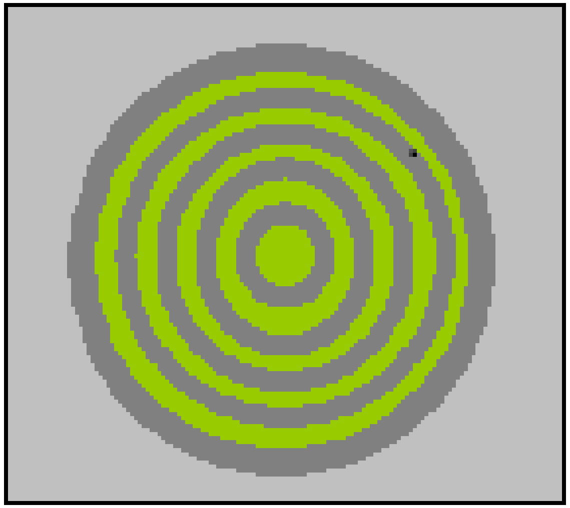
应用Huygens-Fresnel公式，面积分应该由两个部分组成，即屏幕和半无穷大空间的边界。 半无穷大边界面上的积分为零，得到：

$$\varphi(\mathbf{r}) = \frac{-A}{4\pi}\iint_{S_a}\left[\hat{\mathbf{R}}_0\left(\mathrm{j}k+\frac{1}{R_0}\right)+\hat{\mathbf{R}}\left(\mathrm{j}k+\frac{1}{R}\right)\right]\frac{e^{-\mathrm{j}kR_0}}{R_0}\frac{e^{-\mathrm{j}kR}}{R}\cdot \mathrm{d}\mathbf{S}'$$

对于振幅因子，近似认为 $\hat{\mathbf{R}}_0=\hat{\mathbf{R}}$ ，并略去 $1/R$ 高阶项

$$\varphi(\mathbf{r}) = \frac{-\mathrm{j}k}{4\pi}A\iint_{S_a}\left[\hat{\mathbf{R}}_0\cdot\mathbf{n}+\hat{\mathbf{R}}\cdot\mathbf{n}\right]\frac{e^{-\mathrm{j}kR_0}}{R_0}\frac{e^{-\mathrm{j}kR}}{R}\mathrm{d}S'$$

$$= -\frac{\mathrm{j}}{\lambda}A\iint_{S_a}\hat{\mathbf{R}}\cdot\mathbf{n}\frac{e^{-\mathrm{j}kR_0}}{R_0}\frac{e^{-\mathrm{j}kR}}{R}\mathrm{d}S'$$



思考：

- 矢量场惠更斯原理(**Stratton-Chu**公式)及推导？
- 矢量电场和磁场满足的**Sommerfeld**辐射条件？

# 8.1.2 矢量波方程

$$\nabla \times \nabla \times \mathbf{E}_1(\mathbf{r}) - \omega^2 \mu_1 \epsilon_1 \mathbf{E}_1(\mathbf{r}) = i\omega\mu_1 \mathbf{J}(\mathbf{r}). \qquad (8.1.14)$$

$$\nabla \times \nabla \times \mathbf{E}_2(\mathbf{r}) - \omega^2 \mu_2 \epsilon_2 \mathbf{E}_2(\mathbf{r}) = 0. \qquad (8.1.15)$$

$$\nabla \times \nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') - \omega^2 \mu_1 \epsilon_1 \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{I}}\,\delta(\mathbf{r} - \mathbf{r}'), \qquad (8.1.16)$$

$$\nabla \times \nabla \times \overline{\mathbf{G}}_2(\mathbf{r}, \mathbf{r}') - \omega^2 \mu_2 \epsilon_2 \overline{\mathbf{G}}_2(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{I}}\,\delta(\mathbf{r} - \mathbf{r}'). \qquad (8.1.17)$$

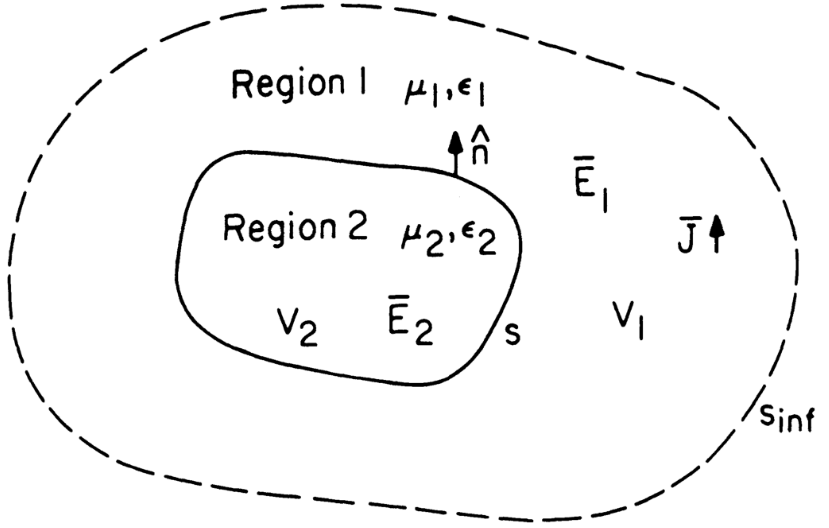



**Figure 8.1.3** A two-region problem where a surface integral equation can be derived.

$$\int_{V_1} dV \; [\nabla \times \nabla \times \mathbf{E}_1(\mathbf{r}) \cdot \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') - \mathbf{E}_1(\mathbf{r}) \cdot \nabla \times \nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')]$$

$$= i\omega\mu_1 \int_{V_1} dV \, \mathbf{J}(\mathbf{r}) \cdot \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') - \mathbf{E}_1(\mathbf{r}'), \qquad \mathbf{r}' \in V_1. \quad (8.1.18)$$

由恒等式：

$$\nabla \cdot \left\{ [\nabla \times \mathbf{E}_1(\mathbf{r})] \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') + \mathbf{E}_1(\mathbf{r}) \times \left[ \nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') \right] \right\}$$

$$= \nabla \times \nabla \times \mathbf{E}_1(\mathbf{r}) \cdot \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') - \mathbf{E}_1(\mathbf{r}) \cdot \nabla \times \nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}'). \quad (8.1.19)$$

$$\mathbf{E}_1(\mathbf{r}') = \mathbf{E}_{inc}(\mathbf{r}') + \int_{S+S_{inf}} dS \, \hat{n} \cdot \left\{ [\nabla \times \mathbf{E}_1(\mathbf{r})] \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') \right.$$

$$\left. + \mathbf{E}_1(\mathbf{r}) \times \nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') \right\}, \quad \mathbf{r}' \in V_1, \quad (8.1.20)$$

其中：

$$\mathbf{E}_{inc}(\mathbf{r}') = i\omega\mu_1 \int_{V_1} dV \, \mathbf{J}(\mathbf{r}) \cdot \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') = i\omega\mu_1 \int_{V_1} dV \, \overline{\mathbf{G}}_1(\mathbf{r}', \mathbf{r}) \cdot \mathbf{J}(\mathbf{r}) \quad (8.1.21)$$

$$\overline{\mathbf{G}}_1^t(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{G}}_1(\mathbf{r}', \mathbf{r}). \quad (8.1.22)$$

进一步变换：

$$\hat{n} \cdot [\nabla \times \mathbf{E}_1(\mathbf{r})] \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') = \hat{n} \times [\nabla \times \mathbf{E}_1(\mathbf{r})] \cdot \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')$$
$$= i\omega\mu_1 \overline{\mathbf{G}}_1(\mathbf{r}', \mathbf{r}) \cdot \hat{n} \times \mathbf{H}_1(\mathbf{r}). \tag{8.1.23}$$

$$\hat{n} \cdot \mathbf{E}_1(\mathbf{r}) \times \nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') = \hat{n} \times \mathbf{E}_1(\mathbf{r}) \cdot \nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')$$
$$= -\left[\nabla \times \overline{\mathbf{G}}_1(\mathbf{r}', \mathbf{r})\right] \cdot \hat{n} \times \mathbf{E}_1(\mathbf{r}), \tag{8.1.24}$$

$$\mathbf{E}_1(\mathbf{r}') = \mathbf{E}_{inc}(\mathbf{r}') + \int_S dS \left\{ i\omega\mu_1 \overline{\mathbf{G}}_1(\mathbf{r}', \mathbf{r}) \cdot \hat{n} \times \mathbf{H}_1(\mathbf{r}) - \left[\nabla \times \overline{\mathbf{G}}_1(\mathbf{r}', \mathbf{r})\right] \cdot \hat{n} \times \mathbf{E}_1(\mathbf{r}) \right\}. \tag{8.1.25}$$

$$\boxed{\left.\begin{array}{ll} \mathbf{r} \in V_1, & \mathbf{E}_1(\mathbf{r}) \\ \mathbf{r} \in V_2, & 0 \end{array}\right\} = \mathbf{E}_{inc}(\mathbf{r}) + \int_S dS' \left\{ i\omega\mu_1 \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') \cdot \hat{n}' \times \mathbf{H}_1(\mathbf{r}') - \left[\nabla' \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')\right] \cdot \hat{n}' \times \mathbf{E}_1(\mathbf{r}') \right\}. \quad (8.1.26)}$$

同理，对于区域２：

$$\left.\begin{array}{ll}\mathbf{r} \in V_2, & \mathbf{E}_2(\mathbf{r}) \\ \mathbf{r} \in V_1, & 0\end{array}\right\} = -\int_S dS' \left\{ i\omega\mu_2 \overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}') \cdot \hat{n}' \times \mathbf{H}_2(\mathbf{r}') - \left[\nabla' \times \overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\right] \cdot \hat{n}' \times \mathbf{E}_2(\mathbf{r}') \right\}, \qquad (8.1.27)$$

由自屏蔽定理建立方程：

$$\mathbf{E}_{inc}(\mathbf{r}) = \int_S dS' \left\{ i\omega\mu_1 \overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}') \cdot \hat{n}' \times \mathbf{H}_1(\mathbf{r}') - \left[\nabla' \times \overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\right] \cdot \hat{n}' \times \mathbf{E}_1(\mathbf{r}') \right\}, \quad \mathbf{r} \in V_2, \qquad (8.1.28a)$$

$$0 = \int_S dS' \left\{ i\omega\mu_2 \overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}') \cdot \hat{n}' \times \mathbf{H}_2(\mathbf{r}') - \left[\nabla' \times \overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\right] \cdot \hat{n}' \times \mathbf{E}_2(\mathbf{r}') \right\}, \quad \mathbf{r} \in V_1. \qquad (8.1.28b)$$

边界条件：

$$\hat{n} \times \mathbf{H}_1(\mathbf{r}) = \hat{n} \times \mathbf{H}_2(\mathbf{r}), \quad \hat{n} \times \mathbf{E}_1(\mathbf{r}) = \hat{n} \times \mathbf{E}_2(\mathbf{r}) \qquad (8.1.29)$$

# 小结：

■ **惠更斯-菲涅尔等效原理**

$$\hat{n}\times\mathbf{H}=\mathbf{J}_s \quad \mathbf{E}\times\hat{n}=\mathbf{M}_s$$

$$\hat{n}\bullet\mathbf{E}=\rho_s/\varepsilon \quad \hat{n}\bullet\mathbf{H}=\rho_s^m/\mu$$

$$\nabla\bullet\mathbf{J}_s=-j\omega\rho_s$$

□标量场Kirchhoff 公式

$$\varphi(\mathbf{r})=\iiint_V G(\mathbf{r},\mathbf{r}')s(r')dV' + \oiint_S\left[G(\mathbf{r},\mathbf{r}')\nabla'\varphi(\mathbf{r}')-\varphi(\mathbf{r}')\nabla'G(\mathbf{r},\mathbf{r}')\right]\cdot\mathrm{d}\mathbf{S}'$$

□矢量场Stratton-Chu公式

$$\mathbf{E}(\mathbf{r})=-j\omega\mu\iiint_V G\mathbf{J}+\frac{1}{k^2}\nabla\bullet\mathbf{J}\nabla'GdV'+\iiint_V\nabla G\times\mathbf{M}dV' -j\omega\mu\oiint_S\left[G(\hat{n}\times\mathbf{H})+\frac{1}{k^2}\nabla\bullet(\hat{n}\times\mathbf{H})\nabla'G\right]\cdot\mathrm{d}\mathbf{S}'+\oiint_S\nabla'G\times(\mathbf{E}\times\hat{n})\cdot\mathrm{d}\mathbf{S}'$$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{inc} - j\omega\mu \oint\!\!\oint_S \left[ G\mathbf{J}_s + \frac{1}{k^2}\nabla\bullet\mathbf{J}_s\nabla' G \right]\cdot d\mathbf{S}' + \oint\!\!\oint_S \nabla' G \times \mathbf{M}_s \cdot d\mathbf{S}'$$

## ■ Sommerfeld辐射条件

### □标量场

$$\lim_{r\to\infty} \phi(\mathbf{r}) = 0$$

$$\lim_{r\to\infty} r\left( \frac{\partial\varphi(\mathbf{r})}{\partial r} - jk\varphi(\mathbf{r}) \right) = 0$$

### □矢量场

$$\lim_{r\to\infty} r\begin{bmatrix} \hat{r}\times\mathbf{E}-\eta\mathbf{H} \\ \mathbf{H}\times\hat{r}-\mathbf{E}/\eta \end{bmatrix} = 0$$

或

$$\lim_{r\to\infty} r\left[ \nabla\times\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} + jk\hat{\mathbf{r}}\times\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} \right] = 0$$

# BACKGROUND-Traditional Formulations [1] [2]

$$\left\{\begin{array}{l} \displaystyle\sum_{l=1}^{2}\frac{a_l}{\eta_l}\,T\text{-}EFIE_l+\sum_{l=1}^{2}b_l\,\text{N-M}FIE_l \\ \displaystyle -\sum_{l=1}^{2}c_l\,N\text{-}EFIE_l+\sum_{l=1}^{2}d_l\,\eta_l\,\text{T-M}FIE_l \end{array}\right.$$

$$T\text{-}EFIE:\left(E_l^s\left(J_l,M_l\right)+E_l^{inc}\right)_{\tan}=\left(E_l\right)_{\tan}$$

$$T\text{-}MFIE:\left(H_l^s\left(J_l,M_l\right)+H_l^{inc}\right)_{\tan}=\left(H_l\right)_{\tan}$$

$$N\text{-}EFIE:n_l\times\left(E_l^s\left(J_l,M_l\right)+E_l^{inc}\right)=n_l\times E_l$$

$$N\text{-}MFIE:n_l\times\left(H_l^s\left(J_l,M_l\right)+H_l^{inc}\right)=n_l\times H_l$$


[1] P. Ylä-Oijala, M. Taskinen, and S. Järvenpää, “Surface integral equation formulations for solving electromagnetic scattering problems with iterative methods,” *Radio Sci.*, vol. 40, no. 6, p. RS6002, 2005.
[2] X. Q. Sheng, J.-M. Jin, J. Song, W. C. Chew, and C.-C. Lu, “Solution of combined-field integral equation using multilevel fast multipole algorithm for scattering by homogeneous bodies,” *IEEE Trans. Antennas Propag.*, vol. 46, no. 11, pp. 1718–1726, 1998.

# BACKGROUND-Traditional Formulations [1] [2]

$$E_l^S\left(J_l, M_l\right) = \eta_l L_l\left(J_l\right) - K_l\left(M_l\right)$$

$$H_l^S\left(J_l, M_l\right) = \frac{1}{\eta_l} L_l\left(M_l\right) + K_l\left(J_l\right)$$

$$L_k\left(X(r)\right) = ik_k \int_S dr' \left[ X(r') + \frac{1}{k_k^2} \nabla' \cdot X(r') \nabla \right] g_k(r, r')$$

$$K_k\left(X(r)\right) = K_{P.V.}^k\left(X(r)\right) + \frac{\Omega_i}{4\pi} I\left(X(r)\right) \times \hat{n}_k$$

[1] P. Ylä-Oijala, M. Taskinen, and S. Järvenpää, "Surface integral equation formulations for solving electromagnetic scattering problems with iterative methods," *Radio Sci.*, vol. 40, no. 6, p. RS6002, 2005.
[2] X. Q. Sheng, J.-M. Jin, J. Song, W. C. Chew, and C.-C. Lu, "Solution of combined-field integral equation using multilevel fast multipole algorithm for scattering by homogeneous bodies," *IEEE Trans. Antennas Propag.*, vol. 46, no. 11, pp. 1718–1726, 1998.

# BACKGROUND-Traditional Formulations [1] [2]

- PMCHWT：

$$\begin{bmatrix} \eta_1 L_1 + \eta_2 L_2 & -\left(K_1 + K_2\right) \\ \left(K_1 + K_2\right) & \dfrac{1}{\eta_1} L_1 + \dfrac{1}{\eta_2} L_2 \end{bmatrix} \begin{bmatrix} J_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} -E_1^{inc} \\ -H_1^{inc} \end{bmatrix}$$

**FEATURES**：Be best in accuracy, but worst in iteration efficiency.

$$\boxed{a_l = \eta_l, b_l = 0 = c_l, d_l = \frac{1}{\eta_l}}$$

- CTF：

$$\begin{bmatrix} L_1 + L_2 & -\left(\dfrac{1}{\eta_1} K_{P.V.}^1 + \dfrac{1}{\eta_2} K_{P.V.}^2\right) \\ \eta_1 K_{P.V.}^1 + \eta_2 K_{P.V.}^2 & L_1 + L_2 \end{bmatrix} \begin{bmatrix} J_1 \\ M_1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & \left(-\dfrac{1}{\eta_1} + \dfrac{1}{\eta_2}\right) \hat{n}_1 \times I \\ \left(\eta_1 - \eta_2\right) \hat{n}_1 \times I & 0 \end{bmatrix} \begin{bmatrix} J_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} -E_1^{inc} / \eta_1 \\ -\eta_1 H_1^{inc} \end{bmatrix}$$

**FEATURES**：An improved version of PMCHWT with a better conditioned matrix, having the identity operator

$$\boxed{a_l = 1, b_l = 0 = c_l, d_l = 1}$$


[1] P. Ylä-Oijala, M. Taskinen, and S. Järvenpää, “Surface integral equation formulations for solving electromagnetic scattering problems with iterative methods,” *Radio Sci.*, vol. 40, no. 6, p. RS6002, 2005.

[2] X. Q. Sheng, J.-M. Jin, J. Song, W. C. Chew, and C.-C. Lu, “Solution of combined-field integral equation using multilevel fast multipole algorithm for scattering by homogeneous bodies,” *IEEE Trans. Antennas Propag.*, vol. 46, no. 11, pp. 1718–1726, 1998.

# BACKGROUND-Traditional Formulations [1] [2]

➢ CNF

$$\begin{bmatrix} \hat{n}_1 \times (K_1 - K_2) & \hat{n}_1 \times \left( \frac{L_1}{Z_1} - \frac{L_2}{Z_2} \right) \\ -\hat{n}_1 \times (Z_1 L_1 - Z_2 L_2) & \hat{n}_1 \times (K_1 - K_2) \end{bmatrix} \begin{bmatrix} J_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} -\hat{n}_1 \times H_1^{inc} \\ \hat{n}_1 \times E_1^{inc} \end{bmatrix}$$

**FEATURES**： Be good in the iteration efficiency, but the worst in accuracy.

$$\boxed{a_l = 0, b_l = 1 = c_l, d_l = 0}$$

➢ JMCFIE

$$\begin{bmatrix} (L_1 + L_2) + \hat{n}_1 \times (K_1 - K_2) & -\left( \frac{1}{\eta_1} K_1 + \frac{1}{\eta_2} K_2 \right) + \hat{n}_1 \times \left( \frac{L_1}{Z_1} - \frac{L_2}{Z_2} \right) \\ (\eta_1 K_1 + \eta_2 K_2) - \hat{n}_1 \times (Z_1 L_1 - Z_2 L_2) & (L_1 + L_2) + \hat{n}_1 \times (K_1 - K_2) \end{bmatrix} \begin{bmatrix} J_1 \\ M_1 \end{bmatrix}$$

$$= \begin{bmatrix} -E_1^{inc} / \eta_1 - \hat{n}_1 \times H_1^{inc} \\ -\eta_1 H_1^{inc} + \hat{n}_1 \times E_1^{inc} \end{bmatrix}$$

**FEATURES**： Performance of JMCFIE is somewhere in between the CTF and the CNF.

$$\boxed{a_l = 1, b_l = 1 = c_l, d_l = 1}$$

[1] P. Ylä-Oijala, M. Taskinen, and S. Järvenpää, “Surface integral equation formulations for solving electromagnetic scattering problems with iterative methods,” *Radio Sci.*, vol. 40, no. 6, p. RS6002, 2005.
[2] X. Q. Sheng, J.-M. Jin, J. Song, W. C. Chew, and C.-C. Lu, “Solution of combined-field integral equation using multilevel fast multipole algorithm for scattering by homogeneous bodies,” *IEEE Trans. Antennas Propag.*, vol. 46, no. 11, pp. 1718–1726, 1998.

# BACKGROUND-Traditional Formulations [1] [2]

- mN-Muller

$$\begin{bmatrix} \hat{n}_1 \times \left( \dfrac{\mu_1}{\mu_1+\mu_2} K_1 - \dfrac{\mu_2}{\mu_1+\mu_2} K_2 \right) & \hat{n}_1 \times \left( \dfrac{\mu_1}{\mu_1+\mu_2} \dfrac{L_1}{Z_1} - \dfrac{\mu_2}{\mu_1+\mu_2} \dfrac{L_2}{Z_2} \right) \\ -\hat{n}_1 \times \left( \dfrac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} Z_1 L_1 - \dfrac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} Z_2 L_2 \right) & \hat{n}_1 \times \left( \dfrac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} K_1 - \dfrac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} K_2 \right) \end{bmatrix} \begin{bmatrix} J_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} -\dfrac{\mu_1}{\mu_1+\mu_2} \hat{n}_1 \times H_1^{inc} \\ \dfrac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} \hat{n}_1 \times E_1^{inc} \end{bmatrix}$$

**FEATURES**: Be best in iteration efficiency, but worst in accuracy.

$$\boxed{\begin{aligned} & a_l = 0 = d_l \\ & b_1 = \mu_1 / (\mu_1 + \mu_2), b_2 = \mu_2 / (\mu_1 + \mu_2) \\ & c_1 = \varepsilon_1 / (\varepsilon_1 + \varepsilon_2), c_2 = \varepsilon_2 / (\varepsilon_1 + \varepsilon_2) \end{aligned}}$$


[1] P. Ylä-Oijala, M. Taskinen, and S. Järvenpää, “Surface integral equation formulations for solving electromagnetic scattering problems with iterative methods,” *Radio Sci.*, vol. 40, no. 6, p. RS6002, 2005.
[2] X. Q. Sheng, J.-M. Jin, J. Song, W. C. Chew, and C.-C. Lu, “Solution of combined-field integral equation using multilevel fast multipole algorithm for scattering by homogeneous bodies,” *IEEE Trans. Antennas Propag.*, vol. 46, no. 11, pp. 1718–1726, 1998.

# BACKGROUND-Traditional Formulations

- Normalized Field Quantities [3]

$$\tilde{E} = \sqrt{\varepsilon}E \quad and \quad \tilde{H} = \sqrt{\mu}H$$

$$\tilde{J} = n \times \tilde{H} = \sqrt{\mu}J$$

$$\tilde{M} = -n \times \tilde{\mathrm{E}} = \sqrt{\varepsilon}M$$

$$\tilde{E}_l^s\left(\tilde{J}_l, \tilde{M}_l\right) = L_l\left(\tilde{J}_l\right) - K_l\left(\tilde{M}_l\right)$$

$$\tilde{H}_l^s\left(\tilde{J}_l, \tilde{M}_l\right) = L_l\left(\tilde{M}_l\right) + K_l\left(\tilde{J}_l\right)$$

$$\Longrightarrow \begin{bmatrix} T(a,b) & n \times T(b,a) \\ -n \times T(c,d) & T(d,c) \end{bmatrix} \begin{bmatrix} \tilde{J} \\ \tilde{M} \end{bmatrix} = \begin{bmatrix} -\tilde{F}(a,b) \\ n \times \tilde{F}(c,d) \end{bmatrix}$$

where $T(a,b) = a \times L_{\tan} + b \times \left(n \times K - I/2\right)$ $\tilde{F}(a,b) = a\left(\tilde{E}^{inc}\right)_{\tan} + bn \times \tilde{H}^{inc}$

[3] M. Taskinen and P. Yla-Oijala, “Current and charge Integral equation formulation,” *IEEE Trans. Antennas Propag.*, vol. 54, no. 1, pp. 58–67, 2006.

# 8.9 体积分方程

## 8.9.1 标量波情况

$$[\nabla^2 + k^2(\mathbf{r})]\phi(\mathbf{r}) = q(\mathbf{r}), \tag{8.9.1}$$

$$[\nabla^2 + k_b^2]g(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'). \tag{8.9.2}$$

$$[\nabla^2 + k_b^2]\phi(\mathbf{r}) = q(\mathbf{r}) - [k^2(\mathbf{r}) - k_b^2]\phi(\mathbf{r}). \tag{8.9.3}$$

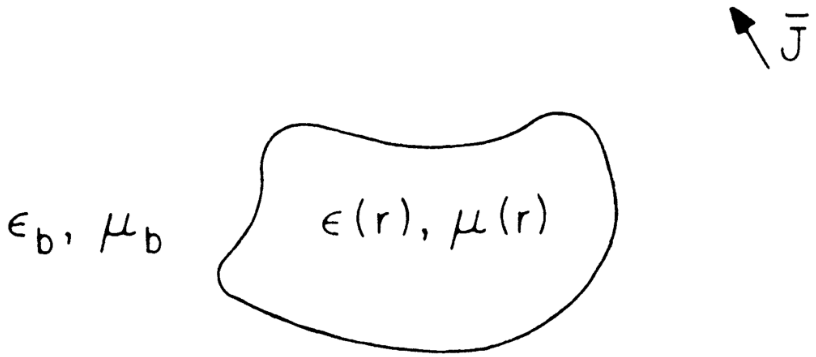



**Figure 8.9.1** A current source radiating in the vicinity of a general inhomogeneity.

$$\phi(\mathbf{r}) = -\int_{V_s} dV' g(\mathbf{r},\mathbf{r}')q(\mathbf{r}') + \int_V dV' g(\mathbf{r},\mathbf{r}')[k^2(\mathbf{r}') - k_b^2]\phi(\mathbf{r}'). \tag{8.9.4}$$

$$\phi(\mathbf{r}) = \phi_{inc}(\mathbf{r}) + \int_V dV' g(\mathbf{r},\mathbf{r}')\boxed{[k^2(\mathbf{r}') - k_b^2]\phi(\mathbf{r}')}. \tag{8.9.5}$$

体等效源

建立体积分方程

$$\phi_{inc}(\mathbf{r}) = \phi(\mathbf{r}) - \int_V dV' g(\mathbf{r},\mathbf{r}')[k^2(\mathbf{r}') - k_b^2]\phi(\mathbf{r}'), \quad \mathbf{r} \in V. \tag{8.9.6}$$

$$\phi_{inc}(\mathbf{r}) = [\mathcal{I} - \mathcal{L}(\mathbf{r},\mathbf{r}')]\phi(\mathbf{r}'), \quad \mathbf{r} \in V, \tag{8.9.7}$$

# 8.9.1 电磁波情况

$$\nabla \times \mu^{-1} \nabla \times \mathbf{E}(\mathbf{r}) - \omega^2 \epsilon \mathbf{E}(\mathbf{r}) = i\omega \mathbf{J}(\mathbf{r}), \tag{8.9.8}$$

$$\nabla \times \mu_b^{-1} \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - \omega^2 \epsilon_b \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \mu_b^{-1} \overline{\mathbf{I}}\, \delta(\mathbf{r} - \mathbf{r}'). \tag{8.9.10}$$

$$\nabla \times (\mu^{-1} - \mu_b^{-1}) \nabla \times \mathbf{E}(\mathbf{r}) - \omega^2 (\epsilon - \epsilon_b) \mathbf{E}(\mathbf{r}) = i\omega \mathbf{J}(\mathbf{r}) - \nabla \times \mu_b^{-1} \nabla \times \mathbf{E}(\mathbf{r}) + \omega^2 \epsilon_b \mathbf{E}(\mathbf{r}), \tag{8.9.9}$$

$$\nabla \times \mu_b^{-1} \nabla \times \mathbf{E}(\mathbf{r}) - \omega^2 \epsilon_b \mathbf{E}(\mathbf{r}) = i\omega \mathbf{J}(\mathbf{r}) + \omega^2 (\epsilon - \epsilon_b) \mathbf{E}(\mathbf{r}) - \nabla \times \left( \frac{1}{\mu} - \frac{1}{\mu_b} \right) \nabla \times \mathbf{E}(\mathbf{r}). \tag{8.9.11}$$

$$\begin{aligned} \mathbf{E}(\mathbf{r}) = i\omega \int_V d\mathbf{r}'\, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mu_b \mathbf{J}(\mathbf{r}') + \omega^2 \int_V d\mathbf{r}'\, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mu_b (\epsilon - \epsilon_b) \mathbf{E}(\mathbf{r}') \\ - \int_V d\mathbf{r}'\, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mu_b \nabla' \times \left( \frac{1}{\mu} - \frac{1}{\mu_b} \right) \nabla' \times \mathbf{E}(\mathbf{r}'). \end{aligned} \tag{8.9.12}$$

体电流

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) + \omega^2 \int_V d\mathbf{r}' \, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \boxed{\mu_b(\epsilon - \epsilon_b)\mathbf{E}(\mathbf{r}')}$$
$$- \int_V d\mathbf{r}' \, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \boxed{\mu_b \nabla' \times \left(\frac{1}{\mu} - \frac{1}{\mu_b}\right) \nabla' \times \mathbf{E}(\mathbf{r}').} \quad (8.9.13)$$

体磁流

对于非磁性媒质

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) + \int_V d\mathbf{r}' \, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot O(\mathbf{r}')\mathbf{E}(\mathbf{r}'), \quad (8.9.14)$$

建立体积分方程

where $O(\mathbf{r}') = \omega^2(\mu\epsilon - \mu_b\epsilon_b) = k^2(\mathbf{r}') - k_b^2.$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) - \overline{\mathcal{L}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}(\mathbf{r}'), \qquad \mathbf{r}' \in V, \quad \mathbf{r} \in V, \quad (8.9.15)$$

$$\mathbf{E}_{inc}(\mathbf{r}) = \left[\overline{\mathcal{I}} - \overline{\mathcal{L}}(\mathbf{r}, \mathbf{r}')\right] \cdot \mathbf{E}(\mathbf{r}'), \qquad \mathbf{r}' \in V, \quad \mathbf{r} \in V, \quad (8.9.16)$$

# 8.10 散射问题的近似解

## 8.10.1 波恩近似

对于非磁性媒质

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) + \int_V d\mathbf{r}' \, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot O(\mathbf{r}')\mathbf{E}(\mathbf{r}'), \tag{8.9.14}$$

where $O(\mathbf{r}') = \omega^2(\mu\epsilon - \mu_b\epsilon_b) = k^2(\mathbf{r}') - k_b^2.$

when $k^2 - k_b^2$ is small,

$$\mathbf{E}(\mathbf{r}) \simeq \mathbf{E}_{inc}(\mathbf{r}). \tag{8.10.1}$$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) + \int_V d\mathbf{r}' \, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot O(\mathbf{r}')\mathbf{E}_{inc}(\mathbf{r}'). \tag{8.10.2}$$

无需求解方程即可计算得到空间中的电磁场分布

波恩近似为单次散射近似，违反能量守恒定律

其中
$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{k_b^2}\right) g(\mathbf{r},\mathbf{r}'). \tag{8.10.3}$$

若散射体尺寸量级为L and $k_b L \ll 1$, $\longleftarrow$ 电小尺寸、低频情形

$$g(\mathbf{r},\mathbf{r}') \sim \frac{1}{L}, \quad \nabla\nabla \sim \frac{1}{L^2}. \tag{8.10.4}$$

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \sim \left(1 + \frac{1}{k_b^2 L^2}\right) \frac{1}{L}, \tag{8.10.5a}$$

$$O(\mathbf{r}) = (k^2 - k_b^2) \sim k_b^2 \Delta\epsilon_r, \tag{8.10.5b}$$

where $\Delta\epsilon_r = \epsilon/\epsilon_b - 1$

$$\int d\mathbf{r}' \sim L^3. \tag{8.10.5c}$$

可见散射场的数量级为：

$$[(k_b L)^2 + 1]\Delta\epsilon_r E_{inc}. \tag{8.10.6}$$

低频波恩近似条件：
$$\boxed{\Delta\epsilon_r \ll 1,} \tag{8.10.7}$$

(1) 若极化电荷效应可忽略,

$$k_b^2 L^2 \Delta\epsilon_r \ll 1. \qquad (8.10.11)$$

低频近似条件在 $\Delta\epsilon_r > 1.$ 情况下仍然成立

(2) 若在绝缘的背景介质中存在非均匀导电媒质, 频率越低，波恩近似性能越差

$$k^2(\mathbf{r}) \sim i\omega\mu\sigma(\mathbf{r}) \qquad k_b^2 = \omega^2\mu\epsilon_b$$

$$\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \sim \frac{1}{\omega^2}, \qquad \omega \to 0. \qquad (8.10.12)$$

$$O(\mathbf{r}) \sim \omega, \qquad \omega \to 0. \qquad (8.10.13)$$

Therefore, the scattered field term in (2) is proportional to $1/\omega$ when $\omega \to 0$.

$$\mathbf{J} = \sigma\mathbf{E} \simeq \sigma\mathbf{E}_{in}, \qquad (8.10.14)$$

the charge $\varrho = \nabla \cdot \mathbf{J}/i\omega$ implying that these charges at

the interface diverge as $1/\omega$ when $\omega \to 0$

若散射体尺寸量级为L, 且 $k_b L >> 1$ ⟵ 电大尺寸、高频情形

$$\nabla\nabla \sim k_b^2. \tag{8.10.8}$$

$$\begin{aligned}\mathbf{E} &\sim e^{i\mathbf{k}_b\cdot\mathbf{r}} e^{i(\mathbf{k}-\mathbf{k}_b)\cdot\mathbf{r}} \\ &\sim \mathbf{E}_{inc} e^{i(\mathbf{k}-\mathbf{k}_b)\cdot\mathbf{r}},\end{aligned} \tag{8.10.9}$$

and hence, $\mathbf{E} \simeq \mathbf{E}_{inc}$ only if $\boxed{(k-k_b)L \ll 1.}$

高频波恩近似条件: $\boxed{k_b L \Delta\epsilon_r \ll 1,}\ k_b L \to \infty.$ (8.10.10)

迭代波恩近似:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) + \int_V d\mathbf{r}' \, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot O(\mathbf{r}')\mathbf{E}(\mathbf{r}'), \tag{8.9.14}$$

where $O(\mathbf{r}') = \omega^2(\mu\epsilon - \mu_b\epsilon_b) = k^2(\mathbf{r}') - k_b^2.$

## 8.10.2 里托夫近似

极化电荷效应不重要时，电磁场边值问题可以由标量波导方程描述

$$[\nabla^2 + k^2(\mathbf{r})]\phi(\mathbf{r}) = 0. \qquad (8.10.15)$$

令

$$\phi(\mathbf{r}) = e^{i\psi(\mathbf{r})}. \qquad (8.10.16)$$

$$\nabla\phi(\mathbf{r}) = i\phi(\mathbf{r})\nabla\psi(\mathbf{r}), \qquad (8.10.17a)$$

$$\nabla\cdot\nabla\phi(\mathbf{r}) = \{i\nabla^2\psi(\mathbf{r}) - [\nabla\psi(\mathbf{r})]^2\}\phi(\mathbf{r}). \qquad (8.10.17b)$$

带入原方程，得到一非线性方程

$$i\nabla^2\psi(\mathbf{r}) - (\nabla\psi)^2 + k^2(\mathbf{r}) = 0. \qquad (8.10.18)$$

根据微扰理论，假定

$$\psi(\mathbf{r}) \sim \psi_0(\mathbf{r}) + \psi_1(\mathbf{r}). \tag{8.10.19}$$

$$i\nabla^2\psi_0(\mathbf{r}) - (\nabla\psi_0)^2 + k_b^2(\mathbf{r}) = 0, \tag{8.10.20}$$

其中$\psi_0(\mathbf{r})$， $\phi_0 = e^{i\psi_0(\mathbf{r})}$. 是波数为$k_b$的背景介质的解

得到

$$i\nabla^2\psi_1(\mathbf{r}) - 2(\nabla\psi_0)\cdot(\nabla\psi_1) - (\nabla\psi_1)^2 + O(\mathbf{r}) = 0, \tag{8.10.21}$$

由恒等式

$$\nabla^2(\phi_0\psi_1) = \psi_1\nabla^2\phi_0 + 2(\nabla\phi_0)\cdot(\nabla\psi_1) + \phi_0\nabla^2\psi_1, \tag{8.10.22}$$

where $\phi_0 = e^{i\psi_0(\mathbf{r})}$. Since $\nabla^2\phi_0 = -k_b^2\phi_0$ and $\nabla\phi_0 = i(\nabla\psi_0)\,\phi_0$, we have

$$\nabla^2(\phi_0\psi_1) = -k_b^2\psi_1\phi_0 + 2i\phi_0(\nabla\psi_0)\cdot(\nabla\psi_1) + \phi_0\nabla^2\psi_1. \tag{8.10.23}$$

得到原波动方程的等价方程

$$\nabla^2(\phi_0\psi_1) + k_b^2\phi_0\psi_1 = -i\phi_0(\nabla\psi_1)^2 + i\phi_0 O(\mathbf{r}). \tag{8.10.24}$$

假设 $\psi_1$ 很小

$$(\nabla^2 + k_b^2)\,\phi_0\psi_1 = i\phi_0 O(\mathbf{r}). \qquad (8.10.25)$$

解得

$$\psi_1(\mathbf{r}) = -\frac{i}{\phi_0(\mathbf{r})}\int d\mathbf{r}'\, g(\mathbf{r},\mathbf{r}')\,\phi_0(\mathbf{r}')O(\mathbf{r}'). \qquad (8.10.26)$$

总场的解为

$$\phi(\mathbf{r}) \simeq \phi_0(\mathbf{r})e^{i\psi_1(\mathbf{r})}. \qquad (8.10.27)$$

里托夫近似条件：

$$(\nabla\psi_1)^2 \ll O(\mathbf{r}). \qquad (8.10.28)$$

若散射体尺寸量级为L and $k_b L \ll 1$, ⟵ 电小尺寸、低频情形

$$\psi_1(\mathbf{r}) \sim k_b^2 L^2 \Delta\epsilon_r, \quad \text{when} \quad k_b L \to 0. \tag{8.10.29}$$

$$\boxed{(k_b L)^2 \Delta\epsilon_r \ll 1,} \tag{8.10.30}$$

若散射体尺寸量级为L, 且 $k_b L >> 1$ ⟵ 电大尺寸、高频情形

$$\phi(\mathbf{r}) \sim e^{i\mathbf{k}\cdot\mathbf{r}} \sim e^{i\mathbf{k}_b\cdot\mathbf{r}} e^{i(\mathbf{k}-\mathbf{k}_b)\cdot r} \sim \phi_0 e^{i\psi_1(\mathbf{r})}. \tag{8.10.31}$$

Therefore, $\psi_1(\mathbf{r}) \simeq (\mathbf{k} - \mathbf{k}_b) \cdot \mathbf{r}$,

$$\psi_1(\mathbf{r}) \sim k_b L \Delta\epsilon_r, \quad k_b L \to \infty. \tag{8.10.32}$$

$$\boxed{\Delta\epsilon_r \ll 1,} \tag{8.10.33}$$

求解散射体内部场时，该里托夫近似比高频时的波恩近似条件要宽松些，但求解散射体外部场时，仍应采用 $\boxed{k_b L \Delta\epsilon_r \ll 1,}$

波恩近似

$$\phi(\mathbf{r}) = \phi_{inc}(\mathbf{r}) + \boxed{\int_V dV' g(\mathbf{r},\mathbf{r}')[k^2(\mathbf{r}') - k_b^2]\phi(\mathbf{r}').} \qquad (8.9.5)$$

里托夫近似

$$\phi(\mathbf{r}) \simeq \phi_0(\mathbf{r}) e^{i\psi_1(\mathbf{r})}. \qquad (8.10.27)$$

$$\phi(\mathbf{r}) \cong \phi_0(\mathbf{r}) + \boxed{i\psi_1(\mathbf{r})\,\phi_0(\mathbf{r}).} \qquad (8.10.35)$$

$$\psi_1(\mathbf{r}) = -\frac{i}{\phi_0(\mathbf{r})}\int d\mathbf{r}'\, g(\mathbf{r},\mathbf{r}')\,\phi_0(\mathbf{r}')O(\mathbf{r}'). \qquad (8.10.26)$$

可见，当散射场很弱时，波恩近似和里托夫近似趋于同一近似

# 习 题

**8.41** (a) For a plane wave at normal incidence on a dielectric slab of thickness $L$, find the exact solution of the reflected wave.

(b) Derive the approximation of the reflected wave when $\frac{\epsilon}{\epsilon_b} - 1 \to 0$, where $\epsilon$ is the permittivity of the dielectric slab and $\epsilon_b$ is the permittivity of the background.

(c) Derive the reflected wave using the Born approximation, and show that this result reduces to that in (b) only if (8.10.10) is satisfied.

**8.42** (a) For a scalar wave equation, show that (8.10.11) is the constraint for the validity of the Born approximation at low frequencies.

(b) Show that the corresponding constraint for two dimensions is

$$k_b^2 L^2 \ln(kL) \Delta\epsilon_r \ll 1,$$