第四章 变分方法

4.1 自伴问题的变分表达式

4.2 非自伴问题的变分表达式

4.3 本征值问题的变分表达式

4.4 基本边界条件和自然边界条件

5.1 线性矢量空间

5.1.1 线性矢量空间

内积：

$$\mathbf{f}^t \cdot \mathbf{g} = \sum_{i=1}^{\infty} f_i \, g_i. \qquad (5.1.2)$$

$$\langle f, g \rangle = \int_a^b dx \, f(x) \, g(x). \qquad (5.1.3)$$

$$\mathbf{f}^\dagger \cdot \mathbf{g} = \sum_{i=1}^{\infty} f_i^* \, g_i \qquad (5.1.4)$$

$$\langle f^*, g \rangle = \int_a^b dx \, f^*(x) \, g(x). \qquad (5.1.5)$$

一个完全的内积空间称为希尔伯特(Hilbert)空间

线性算子：

$$\sum_{j=1}^{N} G_{ij} f_j = h_i, \tag{5.1.11}$$

积分算子：

$$\int_0^a G(x,x')\, f(x')\, dx' = h(x). \tag{5.1.10}$$

$$\mathcal{G} f = h. \tag{5.1.12}$$

微分算子：

$$\left[\frac{d^2}{dx^2} + k^2(x)\right] f(x) = h(x), \quad a < x < b \tag{5.1.13}$$

$$\mathcal{D} f = h, \tag{5.1.14}$$

$$\int_a^b dx'\, \delta(x'-x) \left[\frac{d^2}{dx'^{\,2}} + k^2(x')\right] f(x') = h(x). \tag{5.1.15}$$

$$\sum_{j=1}^{N} \delta_{ij} D_{ij} f_j = h_i. \tag{5.1.16}$$

基函数:

$$f(x)=\sum_{n=1}^{\infty}a_n f_n(x). \tag{5.1.17}$$

正交归一性: $\langle f_n, f_m\rangle=\delta_{nm}.$ (5.1.18)

度量:

$$a_n=\langle f, f_n\rangle, \tag{5.1.19}$$

$$f(x)=\sum_{n=1}^{\infty}f_n(x)\langle f_n, f\rangle. \tag{5.1.20}$$

非正交归一情形:

$$\langle f_m, f\rangle=\sum_{n=1}^{\infty}\langle f_m, f_n\rangle a_n. \tag{5.1.21}$$

defining $B_{mn}=\langle f_m, f_n\rangle$, $c_m=\langle f_m, f\rangle$,

Gram矩阵:

$$c_m=\sum_{n=1}^{\infty}B_{mn}a_n,$$

外积及单位算子：

$$\mathcal{I} = \sum_n g_n\rangle\langle g_n \tag{5.1.26}$$

$$\sum_n \langle f, g_n\rangle\langle g_n, g\rangle = \langle f, g\rangle. \tag{5.1.27}$$

g_n 是正交归一的

bra算符： $\langle f$ 行向量

ket算符： $g\rangle$ 列向量

内积： $\langle f, g\rangle$ 数

外积： $f\rangle\langle g$ 矩阵

状态矢量： f(x) 或 $f\rangle$

$$f\rangle=\sum_{n=1}^{\infty}f_n\rangle\langle f_n,f\rangle=\sum_{n=1}^{\infty}g_n\rangle\langle g_n,f\rangle, \qquad (5.1.30)$$

对于连续量：

$$f\rangle=\int_0^a dx'\,x'\rangle\langle x',f\rangle, \qquad \langle x,f\rangle = f(x) \qquad (5.1.31)$$

$$\langle x,x'\rangle=\delta(x-x'). \qquad (5.1.33)$$

$$\mathcal{I}=\int_0^a dx\,x\rangle\langle x, \qquad (5.1.34)$$

$$\langle f,g\rangle=\int_0^a dx\,\langle f,x\rangle\langle x,g\rangle=\int_0^a dx\,f(x)\,g(x). \qquad (5.1.35)$$

帕萨维尔(Parseval)定理：

$$\langle f^*,g\rangle=\int dx\,\langle f^*,x\rangle\langle x^*,g\rangle=\int dx\,f^*(x)\,g(x)$$

$$=\int dk\,\langle f^*,k\rangle\langle k^*,g\rangle=\int dk\,f^*(k)\,g(k). \qquad (5.1.48)$$

两个状态矢量间的内积不随其表达式不同而变化

转置算子： $\langle f, \mathcal{G} g\rangle = \langle g, \mathcal{G}^t f\rangle.$

伴随算子： $\langle f^*, \mathcal{G} g\rangle = \langle g^*, \mathcal{G}^a f\rangle^*.$

对称算子： $\langle f, \mathcal{G} g\rangle = \langle g, \mathcal{G} f\rangle.$

自伴/厄密算子： $\langle f^*, \mathcal{G} g\rangle = \langle g^*, \mathcal{G} f\rangle^*.$

$$\mathcal{G} f\rangle = h\rangle. \tag{5.1.49}$$

$$f\rangle = \sum_n g_n\rangle\, a_n. \tag{5.1.55}$$

$$\sum_n \langle w_m, \mathcal{G} g_n\rangle\, a_n = \langle w_m, h\rangle. \tag{5.1.57}$$

g_n 基函数

w_m 权函数/测试函数

a_n 待定系数

$\langle w_m, Gg_n\rangle$ 阻抗元素

该求解方法称之为彼得罗夫−伽略金法，或加权留数法，或矩量法

w_m 选为δ函数：点匹配法

w_m 选为g_n：　　伽略金法

本征值问题：

$$\mathcal{G}f_i\rangle = \lambda_i f_i\rangle, \tag{5.1.59}$$

对称：不同本征值对应的本真矢量正交；

$$\text{if } \lambda_i \neq \lambda_j, \text{ then } \langle f_j, f_i\rangle = 0 \text{ when } i \neq j,$$

厄密：本征值是实数，在复内积意义上正交；

$$\langle f_j^*, f_i\rangle = 0$$

厄密算子的本征矢量构成了一个完全的基函数系，而对称算子则不然。
参见作业：5.8

对于线性互易媒质，表示矢量波方程的线性算子是对称的；
对于线性无耗媒质，表示矢量波方程的线性算子是厄密的。
参见作业：5.9

基函数选择原则：

(1)符合问题的边界条件：原点条件、辐射条件、电流连续条件等，余子的边界条件用算子方程去体现；

(2)尽符合未知量分布的实际特点：对称性，振荡性、剖分区域的奇异性，电荷中性等；

(3)形式简单，易于计算。

基函数分类：全域基函数、分域基函数；

旋度共形基函数、散度共形基函数；

标量基函数、矢量基函数；

....

常用基函数：正/余弦基函数、本征基函数、脉冲基函数、三角形基函数、屋脊基函数、RWG基函数、SWG基函数、节点有限元基函数、棱边有限元基函数、高阶插值矢量基函数、高阶叠层矢量基函数

提高数值计算效率的一些方法：

(1)选择形式简单的基、权函数：闭合形式解；

(2)应用某些变换(如FFT)提高内积计算的效率；

(3)迭代方法求解矩阵方程：(如CG迭代、Gmres迭代)，以矩阵-矢量乘积运算替代矩阵求逆运算；

(4)运用加速矩阵-矢量相乘的数值方法：如FFT算法、快速多极(FMA)算法、格林函数插值算法等；

(5)运用矩阵稀疏化算法：如自适应交叉近似(ACA)算法、多层矩阵分解(MLMDA)算法等；

(6)采用混合方法：解析-数值、低频-高频、时域-频域；

5.2 自伴问题的变分表达式

5.2.1 一般概念

$$\mathcal{L}f(\mathbf{r}) = g(\mathbf{r}), \tag{5.2.1}$$

自伴定义：

$$\langle f_2^*, \mathcal{L}f_1\rangle = \langle(\mathcal{L}f_2)^*, f_1\rangle = \langle f_1^*, \mathcal{L}f_2\rangle^*. \tag{5.2.2}$$

对称定义：

$$\langle f_2, \mathcal{L}f_1\rangle = \langle f_1, \mathcal{L}f_2\rangle, \tag{5.2.3}$$

内积定义：

$$\langle f_1, f_2\rangle = \int \mathrm{d}\mathbf{r}\, f_1(\mathbf{r}), f_2(\mathbf{r}). \tag{5.2.4}$$

泛函定义：函数的函数

泛函变分的定义：

$$I = fun(f)$$

$$f = f_e + \delta f, \qquad I = I_e + \delta I \tag{5.2.6}$$

定义泛函：

$$I = \langle f^*, \mathcal{L}f\rangle - \langle f^*, g\rangle - \langle g^*, f\rangle. \tag{5.2.5}$$

可证明：使得式(5. 2. 5)变分为零的点，即驻点，即为方程(5. 2. 1)的解；

提供了一种新的求解算子方程的方法。

证明：

$$I_e+\delta I=\left\langle f_e+\delta f, Lf_e+L\delta f\right\rangle-\left\langle f_e^*+\delta f^*, g\right\rangle-\left\langle g^*, f_e+\delta f\right\rangle$$

$$=\underline{\left\langle f_e, Lf_e\right\rangle}+\boxed{\left\langle f_e, L\delta f\right\rangle}+\boxed{\left\langle \delta f, Lf_e\right\rangle}+\underbrace{\boxed{\left\langle \delta f, L\delta f\right\rangle}}_{\text{二阶误差}}$$

$$-\underline{\left\langle f_e^*, g\right\rangle}-\boxed{\left\langle \delta f^*, g\right\rangle}-\underline{\left\langle g^*, f_e\right\rangle}-\boxed{\left\langle g^*, \delta f\right\rangle}$$

$$\delta I=\langle f_e^*, \mathcal{L}\delta f\rangle+\langle \delta f^*, \mathcal{L}f_e\rangle-\langle \delta f^*, g\rangle-\langle g^*, \delta f\rangle. \quad =0 \qquad (5.2.7)$$

也可对(5)式直接求变分得到

自伴性质： $\langle f_e^*, \mathcal{L}\delta f\rangle=\langle \delta f^*, \mathcal{L}f_e\rangle^*.$ 其中 $Lf_e=g$

精确解 f_e 的泛函的一次变分 δI 为零，表明若f具有一阶误差 δf ，则它的泛函的误差为二阶：二阶精度。

自伴时： $I=\,<f^*, Lf>-2\,\mathrm{Re}<f^*, g>$

同理，若L为对称算子，其相应的变分表达式为：

对称时： $I=\langle f, \mathcal{L}f\rangle-2\langle f, g\rangle. \qquad (5.2.8)$

由于表达式(5)和(8)的二阶性质，仅有一个驻点。

自伴算子：

$$I = \mathbf{a}^{\dagger} \cdot \overline{\mathbf{L}} \cdot \mathbf{a} - \mathbf{a}^{\dagger} \cdot \mathbf{g} - \mathbf{g}^{\dagger} \cdot \mathbf{a} \tag{5.2.10}$$

对称算子：

$$I = \mathbf{a}^{t} \cdot \overline{\mathbf{L}} \cdot \mathbf{a} - 2\mathbf{a}^{t} \cdot \overline{\mathbf{g}}. \tag{5.2.11}$$

矩阵方程：

$$\overline{\mathbf{L}} \cdot \mathbf{a} = \mathbf{g}. \tag{5.2.12}$$

也可定义一个与度量相关的变分表达式：

$$u = \langle h^{*}, f \rangle, \qquad Lf = g \tag{5.2.13}$$

$$\mathcal{L} f_a = h. \tag{5.2.14}$$

自伴算子：

$$u = \frac{\langle f_a^{*}, g \rangle \boxed{\langle h^{*}, f \rangle}}{\langle f_a^{*}, \mathcal{L} f \rangle}. \tag{5.2.15}$$

对称算子：

$$u = \frac{\langle f_a, g \rangle \boxed{\langle f, h \rangle}}{\langle f_a, \mathcal{L} f \rangle}, \tag{5.2.18}$$

用(15)代替(13)计算得到的度量u具有更高阶(二阶)的精度。

5.2.2 瑞利-里兹方法

变分问题求解：

$$f=\sum_{n=1}^{N}a_nf_n, \tag{5.2.19}$$

$$I=\sum_{n=1}^{N}\sum_{n'=1}^{N}a_{n'}^*a_n\langle f_{n'}^*,\mathcal{L}f_n\rangle-\sum_{n=1}^{N}a_n^*\langle f_n^*,g\rangle-\sum_{n=1}^{N}a_n\langle g^*,f_n\rangle. \tag{5.2.20}$$

变为二次型：

$$I=\mathbf{a}^\dagger\cdot\overline{\mathbf{L}}\cdot\mathbf{a}-2\Re e\,[\mathbf{a}^\dagger\cdot\mathbf{g}], \tag{5.2.21}$$

其中：

$$\left[\overline{\mathbf{L}}\right]_{mn}=\langle f_m^*,\mathcal{L}f_n\rangle, \tag{5.2.22}$$

$$[\mathbf{g}]_n=\langle f_n^*,g\rangle, \tag{5.2.23}$$

$$\delta I=\delta\mathbf{a}^\dagger\cdot\overline{\mathbf{L}}\mathbf{a}_0+\mathbf{a}_0^\dagger\cdot\overline{\mathbf{L}}\cdot\delta\mathbf{a}-2\Re e\,[\delta\mathbf{a}^\dagger\cdot\mathbf{g}]=0. \tag{5.2.24}$$

$$2\Re e\left[\delta\mathbf{a}^\dagger\cdot\overline{\mathbf{L}}\cdot\mathbf{a}_0\right]=2\Re e\,[\delta\mathbf{a}^\dagger\cdot\mathbf{g}]. \tag{5.2.25}$$

最终求解矩阵方程：

$$\overline{\mathbf{L}}\cdot\mathbf{a}_0=\mathbf{g}. \tag{5.2.26}$$

变分问题的瑞利-里兹方法与伽略金方法是一致的，伽略金方法的解具有变分稳定性。

对于度量的变分表达式：$u = \langle h^*, f \rangle, \quad Lf = g \qquad (5.2.13)$

$$f = \sum_{n=1}^{N} a_n f_n, \tag{5.2.27}$$

$$f_a = \sum_{m=1}^{N} b_m f_{am}. \qquad Lf_a = h \tag{5.2.28}$$

$$\Rightarrow \quad u = \frac{\sum_{m=1}^{N} \sum_{n=1}^{N} b_m^* a_n \langle f_{am}^*, g \rangle \langle h^*, f_n \rangle}{\sum_{m=1}^{N} \sum_{n=1}^{N} b_m^* a_n \langle f_{am}^*, \mathcal{L} f_n \rangle}. \tag{5.2.29}$$

$$u = \frac{\mathbf{b}^\dagger \cdot \mathbf{g} \mathbf{h}^\dagger \cdot \mathbf{a}}{\mathbf{b}^\dagger \cdot \overline{\mathbf{L}} \cdot \mathbf{a}}, \tag{5.2.30}$$

$$[\mathbf{g}]_m = \langle f_{am}^*, g \rangle,$$

$$[\mathbf{h}]_n = \langle f_n^*, h \rangle,$$

$$[\overline{\mathbf{L}}]_{mn} = \langle f_{am}^*, \mathcal{L} f_n \rangle,$$

$$\Rightarrow \quad \overline{\mathbf{L}} \cdot \mathbf{a}_0 = \mathbf{g}, \tag{5.2.35a}$$

$$\overline{\mathbf{L}} \cdot \mathbf{b}_0 = \mathbf{h}, \tag{5.2.35b}$$

$$\Rightarrow \quad u_0 = \frac{\mathbf{b}_0^\dagger \cdot \mathbf{g} \mathbf{h}^\dagger \cdot \mathbf{a}_0}{\mathbf{b}_0^\dagger \cdot \overline{\mathbf{L}} \cdot \mathbf{a}_0}. \tag{5.2.34}$$

$$u_0 = \mathbf{h}^\dagger \cdot \mathbf{a}_0 = \mathbf{b}_0^\dagger \cdot \mathbf{g}. \tag{5.2.36}$$

(1) 仅需解其中一个方程；(2) 提供了选择基函数和权函数的线索；(3) f_{am} 应能较好地近似辅助问题的解。

5.2.3 对标量波方程的应用

TE： $\phi \to H_z ; p = \to \mu^{-1}$

TM： $\phi \to E_z ; p = \to \varepsilon^{-1}$

$$\nabla \cdot p(\mathbf{r}) \nabla \phi(\mathbf{r}) + k^2(\mathbf{r})\, p(\mathbf{r})\, \phi(\mathbf{r}) = s(\mathbf{r}), \tag{5.2.37}$$

$$\mathcal{L} = \nabla \cdot p \nabla + k^2 p. \tag{5.2.38}$$

无耗介质情形： p, k^2 为实数

自伴条件：

$$\langle \phi_1^*, \nabla \cdot p \nabla \phi_2 \rangle = \langle \phi_2^*, \nabla \cdot p \nabla \phi_1 \rangle^* \tag{5.2.39a}$$

$$\langle \phi_1^*, k^2 p\, \phi_2 \rangle = \langle \phi_2^*, k^2 p\, \phi_1 \rangle^*, \tag{5.2.39b}$$

$$\int_V d\mathbf{r}\, \phi_1^*(\mathbf{r}) \nabla \cdot p \nabla \phi_2(\mathbf{r}) = \boxed{\int_S dS\, \hat{n} \cdot \phi_1^*(\mathbf{r})\, p \nabla \phi_2(\mathbf{r})} - \int_V d\mathbf{r}\, p \nabla \phi_1^*(\mathbf{r}) \cdot \nabla \phi_2(\mathbf{r}). \tag{5.2.41}$$

$\phi_1 = 0$ 或 $\dfrac{\partial \phi_2}{\partial n} = 0$ ，即齐次狄里赫利或纽曼边界条件

满足上述条件时，L为自伴算子

$$I = -\int_V d\mathbf{r}\, p|\nabla\phi(\mathbf{r})|^2 + \int_V d\mathbf{r}\, k^2 p|\phi(\mathbf{r})|^2 - 2\Re e\int_V d\mathbf{r}\, \phi^*(\mathbf{r})\, s(\mathbf{r}). \qquad (5.2.43)$$

变分为零表示能量取极值

有耗介质情形： p, k^2 为复数，L为对称算子

$$I = -\int_V d\mathbf{r}\, p\,(\nabla\phi(\mathbf{r}))^2 + \int_V d\mathbf{r}\, k^2 p\phi^2(\mathbf{r}) - 2\int_V d\mathbf{r}\, \phi(\mathbf{r})\, s(\mathbf{r}). \qquad (5.2.44)$$

度量

$$u = \phi(\mathbf{r}_0) = \langle h^*, \phi(\mathbf{r})\rangle, \qquad (5.2.45)$$

$$h^* = \delta(\mathbf{r} - \mathbf{r}_0).$$

$$\nabla\cdot p(\mathbf{r})\nabla\phi_a(\mathbf{r}) + k^2(\mathbf{r})p(\mathbf{r})\,\phi_a(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_0). \qquad (5.2.46)$$

$$u = \hat{\phi}(\mathbf{r}_0) = \frac{\phi(\mathbf{r}_0)\int_V d\mathbf{r}\, \phi_a^* s}{-\int_V d\mathbf{r}\, p\nabla\phi_a^*\cdot\nabla\phi + \int_V d\mathbf{r}\, k^2 p\,\phi_a^*\phi}. \qquad (5.2.47)$$

5.2.4 对矢量波方程的应用

$$\nabla \times \mu^{-1}\nabla \times \mathbf{E}(\mathbf{r}) - \mu^{-1}k^2\,\mathbf{E}(\mathbf{r}) = i\omega\mathbf{J}(\mathbf{r}), \tag{5.2.48}$$

$$\mathcal{L} = \left(\nabla \times \mu^{-1}\nabla \times\right) - \mu^{-1}k^2. \tag{5.2.49}$$

无耗介质情形： μ, k^2 为实数

自伴条件：

$$\left\langle \mathbf{E}_1^*(\mathbf{r}), \nabla \times \mu^{-1}\nabla \times \mathbf{E}_2(\mathbf{r})\right\rangle = \left\langle \mathbf{E}_2^*(\mathbf{r}), \nabla \times \mu^{-1}\nabla \times \mathbf{E}_1(\mathbf{r})\right\rangle^* \tag{5.2.50a}$$

$$\left\langle \mathbf{E}_1^*(\mathbf{r}), \mu^{-1}k^2\,\mathbf{E}_2(\mathbf{r})\right\rangle = \left\langle \mathbf{E}_2^*(\mathbf{r}), \mu^{-1}k^2\,\mathbf{E}_1(\mathbf{r})\right\rangle^*, \tag{5.2.50b}$$

$$\langle \mathbf{A}, \mathbf{B}\rangle = \int_V d\mathbf{r}\,\mathbf{A}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r}), \tag{5.2.51}$$

$$\int_V d\mathbf{r}\,\mathbf{E}_1^*(\mathbf{r}) \cdot \nabla \times \mu^{-1}\nabla \times \mathbf{E}_2(\mathbf{r}) = \boxed{-\int_S dS\,\hat{n} \cdot \left[\mu^{-1}\mathbf{E}_1^*(\mathbf{r}) \times \nabla \times \mathbf{E}_2(\mathbf{r})\right]} + \int_V d\mathbf{r}\,\mu^{-1}\nabla \times \mathbf{E}_1^*(\mathbf{r}) \cdot \nabla \times \mathbf{E}_2(\mathbf{r}). \tag{5.2.52}$$

$(\hat{n} \times \mathbf{E} = 0)$或 $(\hat{n} \times \nabla \times \mathbf{E} = i\omega\mu\hat{n} \times \mathbf{H} = 0)$ ，即电壁或磁壁边界条件

满足上述条件时，L为自伴算子

$$I=\int_V d\mathbf{r}\,\mu^{-1}|\nabla\times\mathbf{E}(\mathbf{r})|^2-\int_V d\mathbf{r}\,\mu^{-1}k^2\,|\mathbf{E}(\mathbf{r})|^2-2\Re e\left[i\omega\int_V d\mathbf{r}\,\mathbf{E}^*(\mathbf{r})\cdot\mathbf{J}(\mathbf{r})\right]. \tag{5.2.54}$$

变分为零表示能量取极值

有耗介质情形：μ, k^2 为复数，L为对称算子

$$I=\int_V d\mathbf{r}\,\mu^{-1}[\nabla\times\mathbf{E}(\mathbf{r})]^2-\int_V d\mathbf{r}\,\mu^{-1}k^2\,[\mathbf{E}(\mathbf{r})]^2-2i\omega\int_V d\mathbf{r}\,\mathbf{E}(\mathbf{r})\cdot\mathbf{J}(\mathbf{r}). \tag{5.2.55}$$

度量

$$u=\hat{a}\cdot\mathbf{E}(\mathbf{r}_0)=\langle\mathbf{h}^*,\mathbf{E}(\mathbf{r})\rangle, \tag{5.2.56}$$

$$\nabla\times\mu^{-1}\nabla\times\mathbf{E}_a(\mathbf{r})-\mu^{-1}k^2\,\mathbf{E}_a(\mathbf{r})=\hat{a}\delta(\mathbf{r}-\mathbf{r}_0). \tag{5.2.57}$$

$$\hat{a}\cdot\hat{\mathbf{E}}(\mathbf{r}_0)=\frac{i\omega\hat{a}\cdot\mathbf{E}(\mathbf{r}_0)\int_V d\mathbf{r}\,\mathbf{E}_a^*(\mathbf{r})\cdot\mathbf{J}(\mathbf{r})}{\int_V d\mathbf{r}\,\mu^{-1}\nabla\times\mathbf{E}_a^*(\mathbf{r})\cdot\nabla\times\mathbf{E}(\mathbf{r})+\int_V d\mathbf{r}\,\mu^{-1}k^2\,\mathbf{E}_a^*(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})}. \tag{5.2.58}$$

5.3 非自伴问题的变分表达式

5.3.1 一般概念

转置算子：
$$\langle f_1^*, \mathcal{L} f_2\rangle = \langle(\mathcal{L}^a f_1)^*, f_2\rangle. \quad (5.3.1)$$

伴随算子：
$$\langle f_1, \mathcal{L} f_2\rangle = \langle f_2, \mathcal{L}^t f_1\rangle. \quad (5.3.2)$$

$$(\mathcal{L}^t)^* = \mathcal{L}^a.$$

既不对称也不自伴(如非互易媒质)： $\mathcal{L}^a \neq \mathcal{L}$ and $\mathcal{L}^t \neq \mathcal{L}$.

可定义泛函：
$$I = \langle f_a^*, \mathcal{L} f\rangle - \langle f_a^*, g\rangle - \langle g_a^*, f\rangle, \quad (5.3.3)$$

$$\mathcal{L} f = g, \quad \mathcal{L}^a f_a = g_a. \quad (5.3.4)$$

let $f = f_e + \delta f$, $f_a = f_{ae} + \delta f_a$,

$$\delta I = \langle f_{ae}^*, \mathcal{L}\delta f\rangle + \langle \delta f_a^*, \mathcal{L} f_e\rangle - \langle \delta f_a^*, g\rangle - \langle g_a^*, \delta f\rangle. = 0 \quad (5.3.5)$$

若内积不含复共轭：

$$I = \langle f_a, \mathcal{L} f \rangle - \langle f_a, g \rangle - \langle g_a, f \rangle, \tag{5.3.6}$$

$$\mathcal{L} f = g, \quad \mathcal{L}^t f_a = g_a. \tag{5.3.7}$$

对于度量：

$$u = \langle h^*, f \rangle \tag{5.3.8}$$

$$\mathcal{L}^a f_a = h. \tag{5.3.9}$$

$$u = \frac{\langle f_a^*, g \rangle \langle h^*, f \rangle}{\langle f_a^*, \mathcal{L} f \rangle} \tag{5.3.10}$$

5.3.2 瑞利-里兹方法

$$f = \sum_{n=1}^{N} a_n f_n, \quad f_a = \sum_{m=1}^{N} b_m f_{am} \tag{5.3.14}$$

$$I = \sum_{n=1}^{N} \sum_{m=1}^{N} a_n b_m^* \langle f_{am}^*, \mathcal{L} f_n \rangle - \sum_{m=1}^{N} b_m^* \langle f_{am}^*, g \rangle - \sum_{n=1}^{N} a_n \langle g_a^*, f_n \rangle, \tag{5.3.15}$$

$$I = \mathbf{b}^\dagger \cdot \overline{\mathbf{L}} \cdot \mathbf{a} - \mathbf{b}^\dagger \cdot \mathbf{g} - \mathbf{g}_a^\dagger \cdot \mathbf{a}, \tag{5.3.16}$$

$$\delta I = \mathbf{b}_0^\dagger \cdot \overline{\mathbf{L}} \cdot \delta \mathbf{a} + \delta \mathbf{b}^\dagger \cdot \overline{\mathbf{L}} \cdot \mathbf{a}_0 - \delta \mathbf{b}^\dagger \cdot \mathbf{g} - \mathbf{g}_a^\dagger \cdot \delta \mathbf{a}. \tag{5.3.17}$$

$$\overline{\mathbf{L}} \cdot \mathbf{a}_0 = \mathbf{g}, \tag{5.3.18a}$$

$$\overline{\mathbf{L}}^\dagger \cdot \mathbf{b}_0 = \mathbf{g}_a. \tag{5.3.18b}$$

5.3.3 对标量波方程的应用

$$\nabla \cdot p\nabla\phi(\mathbf{r}) + k^2 p\,\phi(\mathbf{r}) = s(\mathbf{r}), \qquad (5.3.20)$$

$$\mathcal{L} = \nabla \cdot p\nabla + k^2 p \qquad (5.3.21)$$

由定义：

$$\mathcal{L}^a = \nabla \cdot p^*\nabla + k^{*2} p^*. \qquad (5.3.22)$$

$$\nabla \cdot p^*\nabla\phi_a(\mathbf{r}) + k^{*2} p^* \phi_a(\mathbf{r}) = s_a(\mathbf{r}). \qquad (5.3.23)$$

$$I = -\int_V d\mathbf{r}\, p\,(\nabla\phi_a)^* \cdot \nabla\phi + \int_V d\mathbf{r}\, k^2 p \phi_a^* \phi - \int_V d\mathbf{r}\, \phi_a^* s - \int_V d\mathbf{r}\, s_a^* \phi, \qquad (5.3.24)$$

5.3.4 对矢量波方程的应用

$$\nabla\times\overline{\mu}^{-1}\cdot\nabla\times\mathbf{E}(\mathbf{r})-\omega^2\,\overline{\epsilon}\cdot\mathbf{E}(\mathbf{r})=i\omega\mathbf{J}(\mathbf{r}). \qquad (5.3.25)$$

$$\mathcal{L}=\left(\nabla\times\overline{\mu}^{-1}\cdot\nabla\times\right)-\omega^2\,\overline{\epsilon}. \qquad (5.3.26)$$

$$\mathcal{L}^a f_a=g_a. \Rightarrow \nabla\times\left(\overline{\mu}^{\dagger}\right)^{-1}\cdot\nabla\times\mathbf{E}_a(\mathbf{r})-\omega^2\overline{\epsilon}^{\dagger}\cdot\mathbf{E}_a(\mathbf{r})=i\omega\mathbf{J}_a(\mathbf{r}), \qquad (5.3.32)$$

$$I=\int_V d\mathbf{r}\,\nabla\times\mathbf{E}_a^*(\mathbf{r})\cdot\overline{\mu}^{-1}\cdot\nabla\times\mathbf{E}(\mathbf{r})-\omega^2\int_V d\mathbf{r}\,\mathbf{E}_a^*(\mathbf{r})\cdot\overline{\epsilon}\cdot\mathbf{E}(\mathbf{r})$$
$$-i\omega\int_V d\mathbf{r}\,\mathbf{E}_a^*(\mathbf{r})\cdot\mathbf{J}(\mathbf{r})+i\omega\int_V d\mathbf{r}\,\mathbf{J}_a^*(\mathbf{r})\cdot\mathbf{E}(\mathbf{r}), \qquad (5.3.31)$$

$$u=\hat{a}\cdot\mathbf{E}(\mathbf{r}_0)=\langle\hat{a}\delta(\mathbf{r}-\mathbf{r}_0),\mathbf{E}(\mathbf{r})\rangle, \qquad (5.3.33)$$

$$\nabla\times\left(\overline{\mu}^{\dagger}\right)^{-1}\cdot\nabla\times\mathbf{E}_a(\mathbf{r})-\omega^2\overline{\epsilon}^{\dagger}\cdot\mathbf{E}_a(\mathbf{r})=\hat{a}\delta(\mathbf{r}-\mathbf{r}_0). \qquad (5.3.34)$$

$$\hat{a}\cdot\hat{\mathbf{E}}(\mathbf{r}_0)=\frac{i\omega\hat{a}\cdot\mathbf{E}(\mathbf{r}_0)\int_V d\mathbf{r}\,\mathbf{E}_a^*(\mathbf{r})\cdot\mathbf{J}(\mathbf{r})}{\int_V d\mathbf{r}\nabla\times\mathbf{E}_a^*(\mathbf{r})\cdot\overline{\mu}^{-1}\cdot\nabla\times\mathbf{E}(\mathbf{r})-\omega^2\int_V d\mathbf{r}\,\mathbf{E}_a^*(\mathbf{r})\cdot\overline{\epsilon}\cdot\mathbf{E}(\mathbf{r})}, \qquad (5.3.35)$$

5.4 本征值问题的变分表达式

5.4.1 一般概念

算子方程：

$$\mathcal{L}f = \lambda \mathcal{B}f \tag{5.4.1}$$

伴随算子：

$$\langle f^*, \mathcal{L}f\rangle - \lambda\langle f^*, \mathcal{B}f\rangle = 0, \tag{5.4.2}$$

$$\lambda = \frac{\langle f^*, \mathcal{L}f\rangle}{\langle f^*, \mathcal{B}f\rangle}. \tag{5.4.2a}$$

$$\langle \delta f^*, \mathcal{L}f_e\rangle + \langle f_e^*, \mathcal{L}\delta f\rangle - \lambda_e(\langle \delta f^*, \mathcal{B}f_e\rangle + \langle f_e^*, \mathcal{B}\delta f\rangle) - \delta\lambda\langle f_e^*, \mathcal{B}f_e\rangle = 0, \tag{5.4.3}$$

转置算子：

$$\langle f, \mathcal{L}f\rangle - \lambda\langle f, \mathcal{B}f\rangle = 0, \tag{5.4.4}$$

$$\lambda = \frac{\langle f, \mathcal{L}f\rangle}{\langle f, \mathcal{B}f\rangle}. \tag{5.4.4a}$$

基于变分表达式的瑞利-里兹求解方法与伽略金方法一致。

对于非自伴问题，定义辅助函数：

$$\mathcal{L}^a f_a = \lambda^* \mathcal{B}^a f_a, \tag{5.4.5}$$

$$\lambda = \frac{\langle f_a^*, \mathcal{L} f\rangle}{\langle f_a^*, \mathcal{B} f\rangle}. \tag{5.4.6}$$

$$\delta\lambda\langle f_{ae}^*, \mathcal{B} f_e\rangle + \lambda_e(\langle \delta f_a^*, \mathcal{B} f_e\rangle + \langle f_{ae}^*, \mathcal{B}\delta f\rangle) = \langle \delta f_a^*, \mathcal{L} f_e\rangle + \langle f_{ae}^*, \mathcal{L}\delta f\rangle. \tag{5.4.7}$$

对于非对称问题，定义辅助函数：

$$\mathcal{L}^t f_a = \lambda \mathcal{B}^t f_a, \tag{5.4.8}$$

$$\lambda = \frac{\langle f_a, \mathcal{L} f\rangle}{\langle f_a, \mathcal{B} f\rangle}. \tag{5.4.9}$$

瑞利-里兹提供了权函数的选择方法。

5.4.2 对标量波方程的应用

$$\nabla \cdot p(\mathbf{r}) \nabla \phi(\mathbf{r}) + \omega^2 p(r)/c^2(\mathbf{r})\, \phi(\mathbf{r}) = 0 \tag{5.4.10}$$

$$\mathcal{L} = \nabla \cdot p \nabla, \quad \mathcal{B} = -\frac{p(\mathbf{r})}{c^2(\mathbf{r})}, \quad \lambda = \omega^2. \tag{5.4.11}$$

无耗情形：自伴算子

$$\omega^2 = \frac{\int_V d\mathbf{r}\, p(\mathbf{r}) |\nabla \phi(\mathbf{r})|^2}{\int_V d\mathbf{r}\, p(\mathbf{r}) |\phi(\mathbf{r})|^2 / c^2(\mathbf{r})}. \tag{5.4.12}$$

有耗情形：对称算子

$$\omega^2 = \frac{\int_V d\mathbf{r}\, p(\mathbf{r}) [\nabla \phi(\mathbf{r})]^2}{\int_V d\mathbf{r}\, p(\mathbf{r}) [\phi(\mathbf{r})]^2 / c^2(\mathbf{r})}. \tag{5.4.13}$$

非互易情形：非自伴算子

5.4.3 对矢量波方程的应用

$$\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}(\mathbf{r}) - \omega^2 \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}(\mathbf{r}) = 0 \qquad (5.4.14)$$

无耗情形：自伴算子

$$\omega^2 = \frac{\int_V d\mathbf{r}\, \nabla \times \mathbf{E}^*(\mathbf{r}) \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}(\mathbf{r})}{\int_V d\mathbf{r}\, \mathbf{E}^*(\mathbf{r}) \cdot \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}(\mathbf{r})}, \qquad (5.4.15)$$

互易情形：对称算子

$$\omega^2 = \frac{\int_V d\mathbf{r}\, \nabla \times \mathbf{E}(\mathbf{r}) \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}(\mathbf{r})}{\int_V d\mathbf{r}\, \mathbf{E}(\mathbf{r}) \cdot \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}(\mathbf{r})}, \qquad (5.4.16)$$

非互易情形：非自伴算子

$$\nabla \times \left(\overline{\boldsymbol{\mu}}^t\right)^{-1} \cdot \nabla \times \mathbf{E}_a(\mathbf{r}) - \omega^2 \overline{\boldsymbol{\epsilon}}^t \cdot \mathbf{E}_a(\mathbf{r}) = 0. \qquad (5.4.17)$$

$$\omega^2 = \frac{\int_V d\mathbf{r}\, \nabla \times \mathbf{E}_a(\mathbf{r}) \cdot \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}(\mathbf{r})}{\int_V d\mathbf{r}\, \mathbf{E}_a(\mathbf{r}) \cdot \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}(\mathbf{r})}. \qquad (5.4.18)$$

5.5 基本边界条件和自然边界条件

自伴条件：

$$\int_V d\mathbf{r}\,\phi_1^*(\mathbf{r})\nabla\cdot p\nabla\phi_2(\mathbf{r}) = \boxed{\int_S dS\,\hat{n}\cdot\phi_1^*(\mathbf{r})\,p\nabla\phi_2(\mathbf{r})} - \int_V d\mathbf{r}\,p\nabla\phi_1^*(\mathbf{r})\cdot\nabla\phi_2(\mathbf{r}). \tag{5.2.41}$$

$\phi_1 = 0$ 或 $\dfrac{\partial\phi_2}{\partial n} = 0$ ，即齐次狄里赫利或纽曼边界条件

$$\int_V d\mathbf{r}\,\mathbf{E}_1^*(\mathbf{r})\cdot\nabla\times\mu^{-1}\nabla\times\mathbf{E}_2(\mathbf{r}) = \boxed{-\int_S dS\,\hat{n}\cdot\left[\mu^{-1}\mathbf{E}_1^*(\mathbf{r})\times\nabla\times\mathbf{E}_2(\mathbf{r})\right]} + \int_V d\mathbf{r}\,\mu^{-1}\nabla\times\mathbf{E}_1^*(\mathbf{r})\cdot\nabla\times\mathbf{E}_2(\mathbf{r}). \tag{5.2.52}$$

$(\hat{n}\times\mathbf{E}=0)$或 $(\hat{n}\times\nabla\times\mathbf{E}=i\omega\mu\hat{n}\times\mathbf{H}=0)$ ，即电壁或磁壁边界条件

对于给定的物理问题，伴随其出现的边界条件，称为基本边界条件；

使得 $\delta I = 0$ 成立的边界条件称为自然边界条件；

在一扩展空间中的f仍可使得变分稳定，降低边界约束，简化求解过程；

在扩展空间中求得的最终解仍然满足原来的基本边界条件。

5.5.1 标量波方程情形

$$\nabla \cdot p(\mathbf{r})\nabla\phi(\mathbf{r}) + k^2 p(\mathbf{r})\,\phi(\mathbf{r}) = s(\mathbf{r}), \quad \mathbf{r} \in V, \tag{5.5.3}$$

$$I = -\int_V d\mathbf{r}\, p\,(\nabla\phi)^2 + \int_V d\mathbf{r}\, k^2 p\,\phi^2 - 2\int_V d\mathbf{r}\,\phi\, s. \tag{5.5.4}$$

$$\delta I = -2\int_V d\mathbf{r}\, p\,(\nabla\phi)\cdot\nabla\delta\phi + 2\int_V d\mathbf{r}\, k^2 p\,\phi\,\delta\phi - 2\int_V d\mathbf{r}\,\delta\phi\, s. \tag{5.5.5}$$

$$\int_V d\mathbf{r}\, p\,(\nabla\phi)\cdot\nabla\delta\phi = \int_S dS\,\hat{n}\cdot(p\nabla\phi\,\delta\phi) - \int_V d\mathbf{r}\,(\nabla\cdot p\nabla\phi)\,\delta\phi. \tag{5.5.6}$$

$$\delta I = 2\int_V d\mathbf{r}\,(\nabla\cdot p\nabla\phi)\,\delta\phi + 2\int_V d\mathbf{r}\,k^2 p\,\phi\,\delta\phi - 2\int_V d\mathbf{r}\,\delta\phi\,s - 2\int_S dS\,\hat{n}\cdot(p\nabla\phi\,\delta\phi). \tag{5.5.7}$$

$$\hat{n}\cdot\nabla\phi = 0, \quad \text{on} \quad S, \tag{5.5.8}$$

$$\mathcal{L} = \nabla\cdot p\nabla - \Delta(S)\,p\,\hat{n}\cdot\nabla + k^2 p. \tag{5.5.9}$$

$$\langle\phi_1, \Delta(S)\,p\,\hat{n}\cdot\nabla\phi_2\rangle = \int_S dS\,\phi_1\,p\,\hat{n}\cdot\nabla\phi_2,$$

$$\langle\phi_1, \mathcal{L}\phi_2\rangle = -\int_V d\mathbf{r}\,(\nabla\phi_1)\cdot p\,(\nabla\phi_2) + \int_V d\mathbf{r}\,k^2 p\,\phi_1\phi_2 \tag{5.5.10}$$

$$\mathcal{L}\phi = \nabla\cdot p\nabla\phi - \Delta(S)p\,\hat{n}\cdot\nabla\phi + k^2 p\,\phi, \tag{5.5.11}$$

$$\nabla\cdot p\nabla\phi - \Delta(S)p\,\hat{n}\cdot\nabla\phi + k^2 p\,\phi = s \tag{5.5.12}$$

$$\mathcal{L} = \nabla\cdot p\nabla - \nabla\cdot p\,\hat{n}\Delta(S) + k^2 p, \tag{5.5.13}$$

$$\begin{aligned}\langle\phi_1, \mathcal{L}\phi_2\rangle = &-\int_V d\mathbf{r}\,(\nabla\phi_1)\cdot p\,(\nabla\phi_2) + \int_S dS\,\hat{n}\cdot(\phi_1 p\nabla\phi_2)\\ &+ \int_S dS\,(\nabla\phi_1)\cdot p\,\hat{n}\phi_2 + \int_V d\mathbf{r}\,k^2 p\,\phi_1\phi_2.\end{aligned} \tag{5.5.14}$$

$$I = -\int_V d\mathbf{r}\, p\,(\nabla\phi)^2 + 2\int_S dS\,\hat{n}\cdot(\phi\, p\nabla\phi) + \int_V d\mathbf{r}\, k^2 p\,\phi^2 - 2\int_V d\mathbf{r}\,\phi\, s. \tag{5.5.15}$$

$$\delta I = 2\int_V d\mathbf{r}\,(\nabla\cdot p\nabla\phi)\,\delta\phi + 2\int_S dS\,\hat{n}\cdot(\phi p\nabla\delta\phi) + 2\int_V d\mathbf{r}\, k^2 p\,\phi\,\delta\phi - 2\int_V d\mathbf{r}\,\delta\phi\, s. \tag{5.5.16}$$

$$I = -\int_V d\mathbf{r}\, p\,(\nabla\phi)^2 + 2\int_S dS\,\beta\,\phi + \int_V d\mathbf{r}\, k^2 p\,\phi^2 - 2\int_V d\mathbf{r}\,\phi\, s. \tag{5.5.17}$$

$$I = -\int_V d\mathbf{r}\, p\,(\nabla\phi)^2 + 2\int_S dS\,\hat{n}\cdot(\phi - \gamma)\, p\nabla\phi + \int_V d\mathbf{r}\, k^2 p\,\phi^2 - 2\int_V d\mathbf{r}\,\phi\, s. \tag{5.5.18}$$

$$I = -\int_V d\mathbf{r}\, p\,(\nabla\phi)^2 + \int_S dS\,\alpha\phi^2 + \int_V d\mathbf{r}\, k^2 p\,\phi^2 - 2\int_V d\mathbf{r}\,\phi\, s. \tag{5.5.19}$$

5.5.2 电磁波情形

$$\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} - \omega^2 \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E} = i\omega \mathbf{J}. \tag{5.5.20}$$

$$\mathcal{L} = \left(\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times\right) - \omega^2 \overline{\boldsymbol{\epsilon}} \tag{5.5.21}$$

$$\mathcal{L}^a = \left[\nabla \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times\right] - \omega^2 \overline{\boldsymbol{\epsilon}}^{\dagger}, \tag{5.5.22}$$

$$\mathcal{L}^t = \left[\nabla \times \left(\overline{\boldsymbol{\mu}}^{t}\right)^{-1} \cdot \nabla \times\right] - \omega^2 \overline{\boldsymbol{\epsilon}}^{t}. \tag{5.5.23}$$

$$\begin{aligned} I = \left\langle \nabla \times \mathbf{E}_a^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} \right\rangle - \omega^2 \langle \mathbf{E}_a^*, \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E} \rangle \\ - i\omega \langle \mathbf{E}_a^*, \mathbf{J} \rangle + i\omega \langle \mathbf{J}_a^*, \mathbf{E} \rangle, \end{aligned} \tag{5.5.24}$$

$$\mathcal{L}^a \mathbf{E}_a = i\omega \mathbf{J}_a, \tag{5.5.25}$$

$$i\omega \mathbf{J}_s = i\omega \hat{n} \times \mathbf{H} = \hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}. \tag{5.5.26}$$

$$\mathcal{L} = \left(\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times\right) - \left[\triangle(S)\, \hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times\right] - \omega^2\, \overline{\boldsymbol{\epsilon}}, \tag{5.5.27}$$

$$\mathcal{L}\mathbf{E} = i\omega \mathbf{J}. \tag{5.5.27a}$$

$$\langle \mathbf{E}_1^*, \mathcal{L}\mathbf{E}_2 \rangle = \left\langle \nabla \times \mathbf{E}_1^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2 \right\rangle - \omega^2 \langle \mathbf{E}_1^*, \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}_2 \rangle. \tag{5.5.28}$$

$$\mathcal{L}^a = \left[\nabla \times \left(\overline{\boldsymbol{\mu}}^\dagger\right)^{-1} \cdot \nabla \times\right] - \left[\triangle(S)\, \hat{n} \times \left(\overline{\boldsymbol{\mu}}^\dagger\right)^{-1} \cdot \nabla \times\right] - \omega^2\, \overline{\boldsymbol{\epsilon}}^\dagger, \tag{5.5.29}$$

$$\langle \mathbf{E}_2^*, \mathcal{L}^a \mathbf{E}_1 \rangle = \left\langle \nabla \times \mathbf{E}_2^*, \left(\overline{\boldsymbol{\mu}}^\dagger\right)^{-1} \cdot \nabla \times \mathbf{E}_1 \right\rangle - \omega^2 \langle \mathbf{E}_2^*, \overline{\boldsymbol{\epsilon}}^\dagger \cdot \mathbf{E}_1 \rangle. \tag{5.5.30}$$

$$\hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} = 0, \quad \text{on} \quad S, \tag{5.5.31a}$$

$$\hat{n} \times \left(\overline{\boldsymbol{\mu}}^\dagger\right)^{-1} \cdot \nabla \times \mathbf{E}_a = 0, \quad \text{on} \quad S, \tag{5.5.31b}$$

$$\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} - \triangle(S)\, \hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} - \omega^2\, \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E} = i\omega\mu\, \mathbf{J}. \tag{5.5.32}$$

$$\mathbf{M}_s = -\hat{n} \times \mathbf{E} \tag{5.5.33}$$

$$\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} - \nabla \times \left[\overline{\boldsymbol{\mu}}^{-1} \cdot \Delta(S)(\hat{n} \times \mathbf{E})\right] - \omega^2 \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E} = i\omega \mathbf{J} \tag{5.5.34}$$

$$\mathcal{L} = \left(\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times\right) - \left[\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \Delta(S)\, \hat{n} \times\right] - \omega^2 \overline{\boldsymbol{\epsilon}}. \tag{5.5.35}$$

$$\begin{aligned}\langle \mathbf{E}_1^*, \mathcal{L}\mathbf{E}_2 \rangle = \left\langle \nabla \times \mathbf{E}_1^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2 \right\rangle - \int_S dS\, \hat{n} \cdot \left(\mathbf{E}_1^* \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2\right) \\ + \int_S dS \left(\nabla \times \mathbf{E}_1^* \cdot \overline{\boldsymbol{\mu}}^{-1}\right) \times \mathbf{E}_2 \cdot \hat{n} - \omega^2 \langle \mathbf{E}_1^*, \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}_2 \rangle.\end{aligned} \tag{5.5.36}$$

$$\mathcal{L}^a = \left(\nabla \times \left(\overline{\boldsymbol{\mu}}^\dagger\right)^{-1} \cdot \nabla \times\right) - \left[\nabla \times \left(\overline{\boldsymbol{\mu}}^\dagger\right)^{-1} \cdot \Delta(S)\, \hat{n} \times\right] - \omega^2 \overline{\boldsymbol{\epsilon}}^\dagger, \tag{5.5.37}$$

$$\begin{aligned}\langle \mathbf{E}_2^*, \mathcal{L}^a \mathbf{E}_1 \rangle = \left\langle \nabla \times \mathbf{E}_2^*, \left(\overline{\boldsymbol{\mu}}^\dagger\right)^{-1} \cdot \nabla \times \mathbf{E}_1 \right\rangle - \int_S dS\, \hat{n} \cdot \left(\mathbf{E}_2^* \times \left(\overline{\boldsymbol{\mu}}^\dagger\right)^{-1} \cdot \nabla \times \mathbf{E}_1\right) \\ + \int_S dS \left[\nabla \times \mathbf{E}_2^* \cdot \left(\overline{\boldsymbol{\mu}}^\dagger\right)^{-1}\right] \times \mathbf{E}_1 \cdot \hat{n} - \omega^2 \langle \mathbf{E}_2^*, \overline{\boldsymbol{\epsilon}}^\dagger \cdot \mathbf{E}_1 \rangle.\end{aligned} \tag{5.5.38}$$

$$\begin{aligned}I &= \langle \mathbf{E}_a^*, \mathcal{L}\mathbf{E} \rangle - i\omega \langle \mathbf{E}_a^*, \mathbf{J} \rangle + i\omega \langle \mathbf{J}_a^*, \mathbf{E} \rangle \\ &= \left\langle \nabla \times \mathbf{E}_a^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} \right\rangle + \left\langle \mathbf{E}_a^*, \Delta(S)\, \hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} \right\rangle \\ &\quad + \left\langle \hat{n} \times \nabla \times \mathbf{E}_a^* \cdot \overline{\boldsymbol{\mu}}^{-1}, \Delta(S)\mathbf{E} \right\rangle - \omega^2 \langle \mathbf{E}_a^*, \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E} \rangle - i\omega \langle \mathbf{E}_a^*, \mathbf{J} \rangle + i\omega \langle \mathbf{J}_a^*, \mathbf{E} \rangle.\end{aligned} \tag{5.5.39}$$

$$\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} - \Delta(S)\,\hat{n} \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} - \omega^2\,\overline{\boldsymbol{\epsilon}} \cdot \mathbf{E} = i\omega\mathbf{J} - i\omega\Delta(S)\,\boldsymbol{\alpha}, \tag{5.5.40a}$$

$$\nabla \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times \mathbf{E}_a - \Delta(S)\,\hat{n} \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times \mathbf{E}_a - \omega^2\,\overline{\boldsymbol{\epsilon}}^{\dagger} \cdot \mathbf{E}_a = i\omega\mathbf{J}_a - i\omega\Delta(S)\,\boldsymbol{\alpha}_a. \tag{5.5.40b}$$

$$I = \left\langle \nabla \times \mathbf{E}_a^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} \right\rangle - \omega^2 \langle \mathbf{E}_a^*, \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E} \rangle - i\omega \langle \mathbf{E}_a^*, [\mathbf{J} - \Delta(S)\boldsymbol{\alpha}] \rangle + i\omega \langle [\mathbf{J}_a^* - \Delta(S)\boldsymbol{\alpha}_a^*], \mathbf{E} \rangle. \tag{5.5.41}$$

$$\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} - \nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \Delta(S)\,\hat{n} \times \mathbf{E} - \omega^2\,\overline{\boldsymbol{\epsilon}} \cdot \mathbf{E} = i\omega\mathbf{J} - \nabla \times \Delta(S)\,\overline{\boldsymbol{\mu}}^{-1} \cdot \boldsymbol{\beta}, \tag{5.5.42a}$$

$$\nabla \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \nabla \times \mathbf{E}_a - \nabla \times \left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \Delta(S)\,\hat{n} \times \mathbf{E}_a - \omega^2\,\overline{\boldsymbol{\epsilon}}^{\dagger} \cdot \mathbf{E}_a = i\omega\mathbf{J}_a - \nabla \times \Delta(S)\left(\overline{\boldsymbol{\mu}}^{\dagger}\right)^{-1} \cdot \boldsymbol{\beta}_a. \tag{5.5.42b}$$

$$I = \langle \mathbf{E}_a^*, \mathcal{L}\mathbf{E} \rangle - i\omega \langle \mathbf{E}_a^*, \mathbf{J} \rangle + \left\langle \nabla \times \mathbf{E}_a^*, \Delta(S)\,\overline{\boldsymbol{\mu}}^{-1} \cdot \boldsymbol{\beta} \right\rangle + i\omega \langle \mathbf{J}_a^*, \mathbf{E} \rangle + \left\langle \Delta(S)\left(\overline{\boldsymbol{\mu}}^{t}\right)^{-1} \cdot \boldsymbol{\beta}_a^*, \nabla \times \mathbf{E} \right\rangle. \tag{5.5.43}$$

$$I = \langle \nabla \times \mathbf{E}_a^*, \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} \rangle - \omega^2 \langle \mathbf{E}_a^*, [\overline{\boldsymbol{\epsilon}} - \Delta(S)\,\overline{\mathbf{r}}] \cdot \mathbf{E} \rangle - i\omega \langle \mathbf{E}_a^*, \mathbf{J} \rangle + i\omega \langle \mathbf{J}_a^*, \mathbf{E} \rangle. \tag{5.5.44}$$

$$\hat{n} \times \mathbf{H} = -i\omega\overline{\mathbf{r}} \cdot \mathbf{E}, \quad \hat{n} \times \mathbf{H}_a = -i\omega\overline{\mathbf{r}}^{\dagger} \cdot \mathbf{E}_a, \tag{5.5.45}$$

习 题

5.5 An arbitrary, square integrable function $f(x)$ defined for $-a < x < a$ can be expanded as a Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} b_n e^{\frac{in\pi x}{a}}.$$

(a) Show that the basis $e^{in\pi x/a}$ is orthogonal with a complex inner product, and derive a new orthonormal basis.

(b) Derive the equivalence of (5.1.42a), and hence, the identity operator (5.1.43) for this basis. Give the explicit expression for $\langle x^*, \mathcal{I}x' \rangle$, the coordinate-space representation of this identity operator.

(c) Derive Parseval's theorem for the Fourier series as in (5.1.48).

5.6 Show that the definitions of symmetric and Hermitian operators given by (5.1.53) and (5.1.54) are similar to those for matrix operators.