Fast root-MUSIC for arbitrary arrays

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By using the manifold separation techniques, root-MUSIC designed for uniform linear arrays has been extended to arbitrary geometries at the cost of increased computational complexity. A fast algorithm is proposed that exploits the Laurent structure of the polynomial to conduct fast spectral factorisation via the Schur algorithm. Then Arnoldi iteration is employed to compute only a few of the largest eigenvalues. This implies that a large number of the unwanted eigenvalues (or roots) are exempt from the calculation and therefore the computational complexity is reduced significantly.

Introduction: Conventional root-MUSIC is under the assumption of uniform linear arrays. However, root-MUSIC has been extended to operate with any arbitrary arrays by using manifold separation techniques (MST) [1, 2]. In [3] an alternative technique, called Fourier-domain root-MUSIC, can also extend the root-MUSIC to arbitrary arrays with improved performance-to-complexity tradeoffs. Unfortunately, these extensions are achieved at the cost of increased computational complexity because the degree of the polynomial is a significantly large number. A computationally efficient method, called line-search root-MUSIC, has been proposed in [3]. Nevertheless this method is essentially identical to the conventional MUSIC, implying that the resolution ability is inferior to the root-based methods [4]. In this Letter, a new algorithm is proposed, which provides the same resolution ability with smaller computational burden.

Extended root-MUSIC: Consider an array of N sensors, with sensor geometry $\boldsymbol{r}$ (planar or linear), operating in the presence of M uncorrelated narrowband sources. By using MST, the manifold vector $\underline{S}(\boldsymbol{r}, \theta)$ can be written as follows:

$$\underline{S}(\boldsymbol{r}, \theta) = \mathbb{G}(\boldsymbol{r})\underline{d}(\theta) + \underline{\varepsilon} \tag{1}$$

where the matrix $\mathbb{G}(\boldsymbol{r}) \in \mathcal{C}^{N\times Q}$ depends on array geometry only. For the details of $\mathbb{G}(\boldsymbol{r})$, see [1, 2]. The Vandermonde structured vector $\underline{d}(\theta) \in \mathcal{C}^{Q\times 1}$ is a function of direction-of-arrival (DOA) only, defined as

$$\underline{d}(\theta) = \frac{e^{j(Q-1/2)\theta}}{\sqrt{2\pi}}[1, z, \ldots, z^{Q-1}]^T \tag{2}$$

where $z = e^{-j\theta}$ and θ is the DOA. $(\cdot)^T$ denotes transpose operation. Note that only the azimuth angle $\theta \in [0^\circ, 360^\circ)$, measured anticlockwise with respect to the x-axis, is considered in this Letter. The modelling error $\underline{\varepsilon}$ can be safely neglected, provided that Q is a sufficiently large number.

Performing eigenvalue decomposition on the covariance matrix of the received data yields $\mathbb{E}_s$ and $\mathbb{E}_n$, which are eigenvectors corresponding to the M largest eigenvalues and the remaining $(N-M)$ eigenvalues, respectively.

Then a polynomial is constructed as follows:

$$\begin{aligned} f(z) &= \underline{S}^H(\boldsymbol{r}, \theta)(\mathbb{E}_n\mathbb{E}_n^H)\underline{S}(\boldsymbol{r}, \theta) \\ &= \underline{d}^H(\theta)\underbrace{(\mathbb{G}^H(\boldsymbol{r})\mathbb{E}_n\mathbb{E}_n^H\mathbb{G}(\boldsymbol{r}))}_{\triangleq\mathbb{A}}\underline{d}(\theta) \\ &= \frac{1}{2\pi}\sum_{i=-(Q-1)}^{Q-1} b_i z^{-i} \end{aligned} \tag{3}$$

where $(\cdot)^H$ denotes conjugate transpose. The coefficient b_i is the sum of entries of $\mathbb{A}$ along the ith diagonal, i.e.

$$b_i = \sum_{\forall m-n=i} A_{mn} \tag{4}$$

with A_{mn} denoting the (m, n)th entry of $\mathbb{A}$. $f(z)$ is a polynomial of degree $(2Q-2)$, meaning that there are $(2Q-2)$ roots. Since $S(\boldsymbol{r}, \theta)$ lies in the signal subspace, the projection of $S(\boldsymbol{r}, \theta)$ onto the noise subspace $\mathbb{E}_n\mathbb{E}_n^H$ is zero, implying that z corresponding to the true DOA is the root of the above polynomial. Therefore, the DOAs can be obtained as the phase angles of the roots closest to the unit circle. This method is referred to as the extended root-MUSIC in this Letter.

However, the requirement of computing all the $(2Q-2)$ roots of (3), coupled with the fact that Q is significantly large, may render this method computationally expensive. Instead, only M roots need to be calculated in the algorithm presented next. Note that the proposed method is also applicable to the Fourier-domain root-MUSIC technique, which yields a polynomial similar to (3).

Proposed algorithm: Taking into account the Hermitian property of $\mathbb{A}$, one obtains $b_i = b_{-i}^*$, where $(\cdot)^*$ represents complex conjugate operation. This implies that $f(z)$ is a Laurent polynomial [5]. Also, $f(z)$ is non-negative because $f(z) = \|\mathbb{E}_n^H\,\underline{S}(\boldsymbol{r}, \theta)\|^2 \geq 0$, where $\|\cdot\|$ denotes the Euclidean norm of a vector. According to Lemma 1 of [5], $f(z)$ can be factorised as

$$f(z) = c_1 f_1(z) f_1^*(1/z^*) \tag{5}$$

where c_1 is a positive constant. From (5), one can observe that the roots of $f(z)$ appear in conjugate reciprocal pairs, i.e. if z_1 is a root of $f(z)$, then $(z_1^{-1})^*$ is also a root. This property suggests that computing half of the roots (i.e. roots of $f_1(z)$) is sufficient to find the roots of interest. To this end, a fast spectral factorisation method based on the Schur algorithm [5] is applied, which can be implemented in the following steps:

1. Initialise a $(Q\times 2)$ matrix $\mathbb{B}_0$, using b_i calculated from (4):

$$\mathbb{B}_0 = \begin{pmatrix} b_0 & b_{-1} & \cdots & b_{-(Q-2)} & b_{-(Q-1)} \\ b_{-1} & b_{-2} & \cdots & b_{-(Q-1)} & 0 \end{pmatrix}^T \tag{6}$$

2. For $k = 1, 2, \cdots$ until convergence, iterate the following steps:
(*a*) $\mathbb{B}_k = \mathbb{B}_{k-1}\mathbb{U}_k$, where $\mathbb{U}_k$ is a (2×2) matrix defined as

$$\mathbb{U}_k = \frac{1}{\sqrt{1-|\gamma|^2}}\begin{pmatrix} 1 & -\gamma \\ -\gamma^* & 1 \end{pmatrix} \tag{7}$$

with $\gamma = [\mathbb{B}_{k-1}]_{1,2}/[\mathbb{B}_{k-1}]_{1,1}$, i.e. the ratio of the two entries of the first row of $\mathbb{B}_{k-1}$.
(*b*) Shift up the second column of $\mathbb{B}_k$ by one element while keeping the first column unaltered.
(*c*) Test for convergence $\|\underline{b}_{1,k} - \underline{b}_{1,k-1}\| <$ threshold, where $\underline{b}_{1,k}$ and $\underline{b}_{1,k-1}$ denote the first column of $\mathbb{B}_k$ and $\mathbb{B}_{k-1}$, respectively. If converged, go to (3), else return to step 2*a*.
3. The coefficients of $f_1(z)$ are $b_{1,k}^*$.

Now the polynomial factor $f_1(z)$, which has all its roots on or inside the unit circle, is obtained. To find the roots, one can construct an unsymmetric companion matrix $\mathbb{M}$ the eigenvalues of which correspond to the roots of $f_1(z)$ (p.348, [6]). Because the eigenvalues of interest must be the largest ones, one can make use of the Arnoldi iteration to calculate only the M largest eigenvalues (pp.499–503, [6]). Note that the function eigs.m of MATLAB has implemented the Arnoldi iteration.

To summarise, the proposed fast root-MUSIC algorithm for arbitrary arrays can be accomplished via the following steps:

1. Compute the sampling matrix $\mathbb{G}(\boldsymbol{r})$. Note that this offline process requires to be done only once for a given array.
2. Form the received data covariance matrix and perform eigenvalue decomposition to obtain the noise subspace $\mathbb{E}_n$ and construct $\mathbb{A}$ in (3). Then the coefficients of $f(z)$ can be calculated from $\mathbb{A}$ using (4).
3. Perform fast spectral factorisation on $f(z)$ via the Schur algorithm to obtain the polynomial factor $f_1(z)$ and the corresponding companion matrix $\mathbb{M}$.
4. Apply the Arnoldi iteration method to calculate the M largest eigenvalues of $\mathbb{M}$. Then DOAs can be estimated by the phase angles of these eigenvalues.

Simulation results: Assume $M = 2$ uncorrelated equally-powered signals impinge on an arbitrary array of $N = 6$ sensors. The signal-to-noise ratio (SNR) is 20 dB. The x-y Cartesian coordinates of the array sensors, in units of half-wavelengths, are given by

$$\boldsymbol{r} = \begin{pmatrix} 0.5, & 0.2, & 0.5, & 1.0, & 1.5, & 2.0 \\ 0, & -0.5, & 0.2, & 0.3, & -0.3, & 0.1 \end{pmatrix}$$

$Q = 43$ is used in the simulations, which provides the modelling error $\|\underline{\varepsilon}\| < 10^{-10}$. Monte Carlo simulations of 1000 trials have been performed in the simulations. Fig. 1 shows the DOA estimation root-mean-square-errors (RMSEs) of three methods (MUSIC, the extended root-MUSIC and the proposed method) against the snapshot number, with DOAs $[\theta_1, \theta_2] = [100^\circ, 105^\circ]$. In Fig. 2, the DOA of the second source varies from 101° to 110°. The two Figures demonstrate that the proposed algorithm provides an asymptotically similar performance in

DOA estimation to the extended root-MUSIC. Also, the proposed algorithm has superior capability to MUSIC when two signal sources are closely spaced or the snapshot number is quite small. This is because root-based methods are immune to radial errors [4]. It is important to point out that the proposed method achieves this performance by calculating only two roots instead of the complete ($2Q - 2 = 84$) roots required by the extended root-MUSIC.

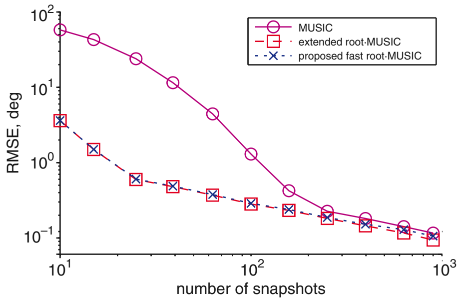

Fig. 1 *DOA estimation RMSEs against snapshot number with SNR = 20 dB ([θ_1, θ_2] = [100°, 105°])*

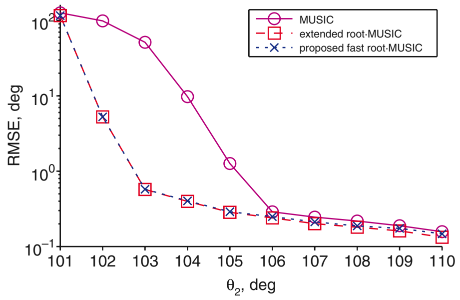

Fig. 2 *DOA estimation RMSEs for $\theta_1 = 100°$ and θ_2 varying from 101° to 110°*

Number of snapshots = 100, SNR = 20 dB

Conclusions: Rather than computing all roots as in the conventional approaches, the proposed fast root-MUSIC algorithm computes only the roots of interest (those corresponding to the true DOAs). Simulation results reveal that the proposed algorithm, with less computational complexity, asymptotically exhibits the same performance in DOA estimation as the extended root-MUSIC.

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One or more of the Figures in this Letter are available in colour online.

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